CWI Quarterly Volume 12 (3&4) 1999, pp. 239 – 314

Dynamics of a Scalar Parabolic Equation

Jack K. Hale
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160
email: hale@math.gatech.edu

In these notes, we consider a scalar parabolic equation which is either autonomous or time dependent. We give a survey of some of the results that deal with the qualitative properties of the dynamics defined by general equations as well as specific results for particular equations. Many of the ideas for a scalar equation have served as a guide to the study of interesting phenomena in systems. We make some remarks about systems, but space limits a detailed discussion of this case. The table of contents describes well the essential plan of the notes.

1. Fundamental concepts

If we consider an evolutionary equation in a Banach space X and assume that the vector field does not depend upon time, then the solution through a point $x \in X$ at time 0 may be represented as $T(t)x$. If each solution is defined for all $t \geq 0$ and there a unique solution through each point, then the family of operators $T(t), t \geq 0$, satisfies the following properties:

$$T(0) = I, \quad T(t + \tau) = T(t)T(\tau). \tag{1.1}$$

A family of transformations $T(t), t \geq 0$, is said to be a *semigroup* of transformations or a *dynamical system* if it satisfies (1.1) and $T(t)x$ is continuous in t, x. We say that $T(t), t \geq 0$, is a C^r-semigroup if it is a semigroup for which $T(t)x$ is C^r in x.

In many situations, the dynamics of a system is defined by the properties of the iterate of a map T on X rather than by an evolutionary equation. In this case, the time is discrete and represented by the integers. We could present both of these cases together by taking a family of transformations indexed by a parameter in a group. However, we present only the continuous case to avoid additional notation and only remark that analogous concepts and results are valid for the discrete case.

For any point $x \in X$, we define the *positive orbit through* x as the set $\gamma^+(x) = \cup_{t\geq 0} T(t)x$. For any set $B \subset X$, we define the *positive orbit through* B as the set $\gamma^+(B) = \cup_{x\in B}\gamma^+(x)$. A set $J \subset X$ is said to be *invariant* if $T(t)J = J$ for $t \geq 0$. Since $T(t)$ maps an invariant set J onto itself, it follows that, for any point $x \in J$ and any $t \geq 0$, there is a $y \in J$ such that $T(t)y = x$. As a consequence, we can define $T(-t)x = y$ and obtain a *negative orbit* $\gamma^-(x)$ *through* x. This corresponds to defining a *backward continuation of an orbit through* x if x belongs to an invariant set J. When we can define a negative orbit through x, we refer to the set $\gamma(x) = \cup_{t\in \mathbb{R}} T(t)x$ as a *globally defined orbit.*

For $x \in X$, we define $\omega(x)$, the ω*-limit set of* x or the ω*-limit set of the orbit through* x, as

$$\omega(x) = \cap_{\tau\geq 0}\mathrm{Cl}\gamma^+(T(\tau)x).$$

This is equivalent to saying that $y \in \omega(x)$ if and only if there is is a sequence $t_n \to \infty$ such that $T(t_n)x \to y$ as $n \to \infty$. In the same way, for any set $B \subset X$, we define $\omega(B)$, the ω*-limit set of* B or the ω*-limit set of the orbit through* B, as

$$\omega(B) = \cap_{\tau\geq 0}\mathrm{Cl}\gamma^+(T(\tau)B).$$

This is equivalent to saying that $y \in \omega(B)$ if and only if there are sequences $t_n \to \infty$, $x_n \in B$, such that $T(t_n)x_n \to y$ as $n \to \infty$.

We remark that $\omega(B)$ may not be $\cup_{x\in B}\omega(x)$. In fact, consider the simple example, $\dot{x} = x - x^3$ and the set B to be the interval $[-2, 2]$. Then $\omega(B) = [-1, 1]$ and $\cup_{x\in B}\omega(x) = \{0, \pm 1\}$.

We say that a set A *attracts* a set B under $T(t)$ if $\lim_{t\to\infty} \mathrm{dist}\,(T(t)B, A) = 0$, where $\mathrm{dist}\,(B, A) = \sup_{x\in B} \mathrm{dist}\,(x, A)$. A basic but very elementary result is the following

Lemma 1.1. *If $B \subset X$ and $\mathrm{Cl}\,\gamma^+(B)$ is compact, then $\omega(B)$ is compact, invariant, and $\omega(B)$ attracts B. Furthermore, if B is connected, then so is $\omega(B)$.*

In the case where $X = \mathbb{R}^n$, the hypotheses of Lemma 1.1 are satisfied if we only assume that $\gamma^+(B)$ is bounded since the closure of bounded sets in $\mathbb{R}^n$ are compact.

If there is a $t_0 \geq 0$ such that $T(t)$ is compact for $t \geq t_0$, then the hypotheses of Lemma 1.1 also are satisfied if we assume only that $\gamma^+(B)$ is bounded. In this case, the unit ball in X is not compact if X is infinite dimensional, but this inconvenience is overcome by the fact that the mapping $T(t)$ is compact for $t \geq t_0$. Semigroups which are compact for $t \geq t_0$ occur in many applications including, for example, reaction diffusion equations and retarded differential difference equations.

There are many other applications for which the semigroup is not compact for any $t \geq 0$ and yet the conclusion of Lemma 1.1 is true. This is the case, for example, for the wave equation with appropriate dissipation and certain types of neutral differerntial differential equations. Even though these more complicated equations are not the primary purpose of these notes, we introduce the more general class of semigroups for which Lemma 1.1 is true.

Motivated by the fact that $T(t)\gamma^+(B) \subset \gamma^+(B)$ for all $t \geq 0$ and that $\omega(B)$ attracts B is equivalent to $\omega(B)$ attracts $\gamma^+(B)$, we make the following definition. We say that the semigroup $T(t), t \geq 0$ is *asymptotically smooth* if, for any bounded set B in X with $T(t)B \subset B$ for $t \geq 0$, there exists a compact set J in the closure of B such that J attracts B. The above definition was introduced by Hale, LaSalle and Slemrod (1972) and is equivalent to the definition of asymptotically compact introduced by Ladyzenskaya (1987).

We notice that, if $T(t)$ is compact and bounded for $t \geq t_0$, then $T(t), t \geq 0$, is asymptotically smooth. Also, if

$$T(t) = S(t) + U(t), \tag{1.2}$$

where $U(t)$ is compact for $t \geq 0$ and there are continuous functions $k(r) > 0$, $\alpha(r) > 0$, $r \geq 0$, such that

$$|S(t)x| \leq k(r)e^{-\alpha(r)t}, \quad t \geq 0, \quad |x| \leq r, \tag{1.3}$$

then $T(t)$ is asymptotically smooth. For each $t > 0$, the operator $T(t)$ may not be compact, but as $t \to \infty$ it becomes compact. The semigroup $T(t)$ also is asymptotically smooth if there are positive constants k, α and a precompact pseudo metric $\rho(t, \cdot, \cdot)$ such that

$$|T(t)x - T(t)y| \leq ke^{-\alpha t}|x - y| + \rho(t, x, y)$$

for all $t \geq 0$, $x, y \in X$ (see Hale (1988)).

Lemma 1.2. *If $T(t)$ is asymptotically smooth and $\gamma^+(B)$ is bounded, then the conclusions of Lemma 1.1 are true.*

The next important concepts deal with attractors. A set $\mathcal{A}$ is said to be a *minimal global attractor* if $\mathcal{A}$ is invariant, attracts bounded sets and is minimal with respect to this property (Ladyzenskaya (1987)). If the global attractor is compact, we will refer to it as the *compact global attractor*.

Lemma 1.3. *If $T(t)$ is asymptotically smooth and, for each bounded set B, $\gamma^+(B)$ is bounded, then there exists a minimal global attractor $\mathcal{A}$, it is connected and is given by*

$$\mathcal{A} = \mathrm{Cl} \cup_{B \subset X} \omega(B).$$

Contrary to an assertion of Hale and Raugel (Equadiff 1992), the attractor $\mathcal{A}$ may not be locally compact as shown by example of Volera (1997).

For the example $\dot{x} = -\varepsilon x, \dot{y} = -y$, the minimal global attractor is the x-axis for $\varepsilon = 0$ and the compact global attractor is the origin for $\varepsilon > 0$. In this example, a small dissipation in the x-direction turned the minimal global attractor into a compact global attractor.

We say that $T(t)$ is *point dissipative* if there is a bounded set B in X such that, for any $x \in X$, there is a $t_0 = t_0(x, B)$ such that $T(t)x \in B$ for $t \geq t_0$. An *equilibrium point* of $T(t)$ is a point x such that $T(t)x = x$ for all $t \in \mathbb{R}$. We denote the set of equilibrium points of $T(t)$ by E.

THEOREM 1.1. *If $T(t)$ is point dissipative and there is an $r \geq 0$ such that $T(t)$ is compact and bounded for $t \geq r$, then there exists the compact global attractor $\mathcal{A}$ and it is connected. Furthermore, if $T(t)$ satisfies (1.2) for $t \geq 0$, then the set E of equilibrium points is nonempty.*

Theorem 1.1 can be useful in many situations; in particular, to ordinary differential equations, reaction diffusion systems and retarded functional differential equations with finite delay.

For the case where $T(t)$ is asymptotically smooth, we need an additional condition to ensure the existence of the global attractor.

THEOREM 1.2. *If $T(t)$ is asymptotically smooth, $\gamma^+(B)$ is bounded if B is bounded, and $T(t)$ is point dissipative, then there exists the compact global attractor $\mathcal{A}$ and it is connected. Furthermore, if $T(t)$ satisfies (1.2) for $t \geq 0$, then the set E of equilibrium points is nonempty.*

Theorem 1.2 can be very useful in the discussion of partial differential equations for which the family of maps $T(t)$ is a group; for example, the damped wave equation and the Fitzhugh-Nagumo equation. It also is appropriate for some neutral functional differential equations and retarded equations with infinite delay.

REMARK 1.1. Theorems 1.1 and 1.2 are valid for discrete dynamical systems defined by a map T. Of course, the assertion about equilibrium points is replaced by fixed points of T. If the discrete dynamical system arises as the Poincaré map of an evolutionary equation with coefficients periodic in the time variable, then fixed points of the Poincaré map correspond to periodic solutions of the evolutionary equation of the same period as the vector field. The existence of a compact global attractor for the Poincaré map then implies the existence of such a periodic solution.

It is difficult to trace the earliest papers which contained discussions involving concepts related to global attractors. The idea is very old and the interested reader can consult BABIN and VISHIK (1991), HALE (1988), HALE, MAGALHÃES and OLIVA (1984), HALE and VERDUYN-LUNEL (1993), HARAUX (1989), LADYZENSKAYA (1987), TÉMAM (1988) and the references therein.

Let $c(\mathcal{A})$ be the *capacity* of the set $\mathcal{A}$ and let $\dim_H(\mathcal{A})$ be the Hausdorff dimension of $\mathcal{A}$.

Theorem 1.3. *If X is a Banach space, $T(t) : X \to X$ is a C^1-semigroup satisfying (1.2), (1.3), is point dissipative and positive orbits of bounded sets are bounded, then the compact global attractor $\mathcal{A}$ has the following properties:(1) $c(\mathcal{A}) < \infty$.(2) If $d = 2c(\mathcal{A}) + 1$ and S is any linear subspace of X with $\dim S \geq d$, then there is a residual set Π of the space of all continuous projections P of X onto S (taken with the uniform operator topology) such that $P|\mathcal{A}$ is one-to-one for every $P \in \Pi$.*

For compact maps and X a Hilbert space, Mallet-Paret (1976) proved that $\dim_H(\mathcal{A}) < \infty$. The result as stated is due to Mañé (1981). Having the capacity finite implies that the Hausdorff dimension is finite. The second part of the theorem says that the compact global attractor can be unraveled in a residual set of directions onto a finite dimensional subspace of large dimension.

The semigroup $T(t)$ is *gradient* if (i) $\gamma^+(x)$ bounded implies that $\mathrm{Cl}\,(\gamma^+(x))$ is compact, (ii) There exists a continuous Lyapunov function $V : X \to \mathbb{R}$; that is, $V(T(t)x)$ is nonincreasing in t for each x and, if $V(T(t)x) = V(x)$ for all $t \in \mathbb{R}$, then $x \in E$, the set of equilibrium points.

If, in addition, $V(x) \to \infty$ as $|x| \to \infty$, then each positive orbit is bounded.

The simplest example of a gradient system is the ODE $\dot{y} = -\nabla F(y)$ with the Lyapunov function $V(y) = F(y)$.

Another interesting example is the parabolic PDE

$$\begin{aligned} u_t - \Delta u &= f(u) \quad \text{in}\, \Omega, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{in}\, \partial\Omega, \end{aligned} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ and f is a C^1-function satisfying a growth condition which will ensure that the initial value problem is well defined in $H^1(\Omega)$. If we let $F(u) = -\int_0^u f(s)ds \to \infty$ as $|u| \to \infty$ and $V(\varphi) = \int_0^1 [\frac{1}{2}(\partial_x \varphi)^2(x) + F(\varphi(x))]dx$, then $V(\varphi) \to \infty$ as $|\varphi| \to \infty$. Also, for smooth initial data,

$$\dot{V}(u(t,\cdot)) = -\int_0^1 [\partial_t u(t,x)]^2 dx \leq 0. \tag{1}$$

A density argument shows that

$$V(u(t,\cdot)) = V(u(0,\cdot)) - \int_0^t \int_0^1 [\partial_t u(\tau,x)]^2 dx d\tau.$$

This equality shows that solutions are defined for all $t \geq 0$ and that $\gamma^+(\varphi)$ is bounded for each $\varphi \in H^1(\Omega)$. The semigroup defined by (1.4) is compact for $t > 0$ and therefore each bounded orbit must have a compact, invariant ω-limit set. Relation (1.5) implies that V is constant on the ω-limit set and also that $u_t = 0$ for each solution on the ω-limit set. We therefore conclude that V is a Lyapunov function and (1.4) is gradient.

For the linearly damped hyperbolic equation

$$
\begin{aligned}
u_{tt} + \beta u_t - \Delta u &= f(u) \quad \text{in}\, \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{in}\, \partial\Omega,
\end{aligned}
\tag{1.6}
$$

with $\beta > 0$ constant and appropriate growth conditions on f, it can be shown that this system is gradient in $H^1(\Omega) \times L^2(\Omega)$. The Lyapunov function is $V(\varphi, \psi) = \int_0^1 [\frac{1}{2}(\partial_x \varphi(x))^2 + \frac{1}{2}\psi^2(x) + F(\varphi(x))]dx$ and

$$
\dot{V}(u(t,\cdot), \partial_t u(t,\cdot)) = -\int_0^1 [\partial_t u(t,x)]^2 dx \leq 0.
$$

Furthermore, the semigroup generated by (1.6) satisfies (1.2), (1.3). If f is a compact map from $H^1(\Omega)$ to $L^2(\Omega)$, then $S(t)$ is the semigroup generated by the linear equation (1.6) with $f \equiv 0$ and is verified by using the variation of constants formula (see, for example, Hale (1988)). If this map is not compact, then the construction of the map $S(t)$ is more complicated (see Arrieta, Carvalho and Hale (1992)).

If $T(t)$ is gradient and there is a compact global attractor $\mathcal{A}$ for which each equilibrium point is hyperbolic, then

$$
\mathcal{A} = \cup_{\varphi \in E} W^u(\varphi),
$$

where $W^u(\varphi)$ is the unstable manifold of φ.

In the applications, it is of interest to know when an orbit of a gradient system is *convergent*; that is, when does the ω-limit set $\omega(\varphi)$ consist of a single point in the set E? For each of the systems mentioned above as well as much more general systems, it is enough to know that there is a $\psi \in \omega(\varphi)$ with the property that the linear variational equation about ψ has zero as an eigenvalue of multiplicity at most one (Hale and Raugel (1992)).

A *Morse decomposition* for the compact global attractor $\mathcal{A}$ is a finite collection $\{M_i\}_{i=1}^k$ of mutually disjoint compact invariant sets (referred to as the *Morse sets*) such that, for any $\varphi \in \mathcal{A}$, one of the following possibilities holds:

(1) there exists $1 \leq i \leq k$ such that $\omega(\varphi) \subset M_i$,
(2) there exist $1 \leq i < j \leq k$ such that $\alpha(\varphi) \subset M_i$, $\omega(\varphi) \subset M_j$.

If we have a Lyapunov function V for a gradient system for which there exists a compact global attractor and all equilibrium points are hyperbolic, then there is a simple way to obtain an interesting Morse decomposition. There is an integer k such that the set $V|E = \{V_1 > V_2 > \cdots > V_k\}$ for some constants V_i. The set M_i can be chosen as $M_i = \{\varphi : V(\varphi) = V_i\}$.

Another important concept in dynamical systems is a Morse-Smale system. We define this only for a discrete dynamical system π since it is all that is needed in the text. A point $\varphi \in X$ is said to be a *nonwandering point of* π

if, for any neighborhood U of φ and any nonnegative integer N, there exist a $\psi \in U$ and an $n \geq N$ such that $\pi^n \psi \in U$. Following Oliva (see Hale, Magalhães and Oliva (1984)), we say that the discrete dynamical system defined by π is *Morse-Smale* if there is a compact global attractor $\mathcal{A}_\pi$ of π, $\pi|\mathcal{A}_\pi$ and its derivative are injective, the nonwandering set consists of a finite number of hyperbolic periodic points and their stable and unstable manifolds intersect transversally. Morse-Smale systems are *structurally stable* when restricted to the attractor; that is, there is a neighborhood U of π in the C^1-Whitney topology such that, for any $\tilde{\pi} \in U$, there is a homeomorphism $h : \mathcal{A}_\pi \to \mathcal{A}_{\tilde{\pi}}$ which takes orbits of π to orbits of $\tilde{\pi}$ and preserves the sense of direction in discrete time.

2. Autonomous - separated boundary conditions

2.1. Gradient structure

In this section, we consider the equation

$$u_t = u_{xx} + f(x, u, u_x) \quad \text{in } \Omega = (0,1), \tag{2.1.1}$$

with the boundary conditions

$$\alpha u_x + \beta u|_{x=0} = 0 = \gamma u_x + \delta u_x|_{x=1}, \tag{2.1.2}$$

where $\alpha, \beta, \gamma, \delta$ are constants which can be normalized so that $\alpha^2 + \beta^2 = 1 = \gamma^2 + \delta^2$. The function f is assumed to be C^2.

We assume that there is an s such that (2.1.1), (2.1.2) defines a local semigroup $T(t)$ on $H^s(0,1)$ and that, if B is a bounded set in $H^s(0,1)$, then the closure of $T(t)B$ is compact if $t > 0$. If f is independent of u_x and is a C^2-function, we can take $s = 1$. If f depends upon u_x, s depends on the growth rate of u_x for large u_x. If the growth rate is less than cubic, then we can take $s = 2$.

We also suppose that the solutions of the initial value problem for the ordinary differential equation

$$y'' = f(x, y, y'), \quad y(x_0) = y_0,\ y'(x_0) = y_0', \tag{2.1.3}$$

are defined for $0 \leq x \leq 1$, $0 \leq x_0 \leq 1$, $y_0 \in \mathbb{R}$, $y_0' \in \mathbb{R}$.

The following result is due to Zelenyak (1968). Matano (1978) also proved the same result for the special case when $f = f(x,u)$ does not depend upon u_x.

Theorem 2.1.1. *System (2.1.1) is gradient. Furthermore, if $\gamma^+(\varphi)$ is a bounded orbit, then $\omega(\varphi)$ is a singleton.*

Zelenyak (1968) actually proved this result for a more general system and even allowed nonlinear boundary conditions. The more general system is given explicitly as

$$u_t = a(x, u, u_x)u_{xx} + f(x, u, u_x) \quad \text{in } \Omega = (0,1), \tag{2.1.4}$$

with the boundary conditions

$$\alpha(u_x + \varphi(u)) + \beta u|_{x=0} = 0 = \gamma(u_x + \psi(u)) + \delta u_x|_{x=1}, \tag{2.1.5}$$

where $\alpha, \beta, \gamma, \delta$ are constants which can be normalized so that $\alpha^2 + \beta^2 = 1 = \gamma^2 + \delta^2$, there is a constant $a_0 > 0$ such that $a(x, u, u_x) \geq a_0$ and the functions a, f, φ, ψ are C^3. Of course, it is necessary also to assume the solutions of the initial value problem for the ordinary differential equation

$$a(x, y, y')y'' = f(x, y, y'), \quad y(x_0) = y_0,\ y'(x_0) = y_0', \tag{2.1.6}$$

are defined for $0 \leq x \leq 1$, $0 \leq x_0 \leq 1$, $y_0 \in \mathbb{R}$, $y_0' \in \mathbb{R}$.

The first step of the proof is to show that there is a Lyapunov function for (2.1.1).

Lemma 2.1.1. There exist C^2 functions $\rho(x, \xi, \eta) > 0$, $\Phi(x, \xi, \eta)$ such that, if

$$V(\varphi) = -\int_0^1 \Phi(x, u, u_x)dx, \tag{2.1.7}$$

and $u(t, x)$ is a solution of (2.1.1), (2.1.2), then

$$V(u(t)) = V(u(0)) - \int_0^t \int_0^1 \rho(x, u, u_x)[u_\tau]^2 dx d\tau. \tag{2.1.8}$$

The proof of this result is rather technical but the idea is to assume that (2.1.8) is satisfied, formally differentiate to obtain a partial differential equation involving Φ and ρ and show that the resulting equation has a solution of the type described in the lemma.

If we now assume that $\gamma^+(\varphi)$ is bounded, then it is compact and has an ω-limit set $\omega(\varphi)$. Since $V(u(t))$ is nonincreasing in t, the function V will be constant on $\omega(\varphi)$ and therefore (2.1.8) implies that we must have $u_t = 0$ on $\omega(\varphi)$; that is, $\omega(\varphi) \subset E$, the set of equilibrium points.

To show that $\omega(\varphi)$ is a singleton in E, we can use the general result of Hale and Raugel (1992) mentioned in Chapter 1. In fact, it follows from the Sturm-Liouville theory that the eigenvalues of the linear variational equation about an equilibrium point are simple. In particular, if 0 is an eigenvalue, then it must be simple and the general theorem of Hale and Raugel (1992) on convergence applies.

Remark 2.1.1. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and consider the system

$$\begin{aligned} u_t - \Delta u &= f(x, u, \nabla u) \quad \text{in}\, \Omega, \\ \alpha u + \beta \frac{\partial u}{\partial x} &= 0 \quad \text{in}\, \partial\Omega. \end{aligned} \tag{2.1.9}$$

In general, system (2.1.9) is not gradient. On the other hand, if the ω-limit set $\omega(\varphi)$ of an orbit contains only equilibrium points and there is a point

$\psi \in \omega(\varphi)$ which is linearly stable, then $\omega(\varphi)$ is a singleton. This follows from the convergence result of Hale and Raugel (1992) since the linear variational equation about ψ has the eigenvalue with maximum real part simple and real. Since we assume linear stability of ψ, it follows that, if 0 is an eigenvalue, it must be simple.

System (2.1.9) satisfies a maximum principle. As a consequence, the solution operator is strongly monotone and it follows that most bounded orbits have ω-limit sets which consist of precisely one equilibrium point and it is linearly stable (see SMITH (1995) for precise statements and historical references). On the other hand, there exist $f(x, u, \nabla u)$ such that equation (2.1.9) has chaotic solutions (POLÁČIK (1991),(1995), POLÁČIK and RYBAKOWSKI (1995)). These motions will be unstable but they do exist. If $f = f(x, u)$, this system is a gradient system (see, for example, MATANO (1979)) and therefore the ω-limit set of any bounded orbit belongs to the set of equilibrium points. However, there exists a function $f(x, u)$ such that, for Ω the unit ball in $\mathbb{R}^2$, there is an orbit whose ω-limit set is a complete circle of equilibrium points (POLÁČIK and RYBAKOWSKI (1996)). Of course, this set of equilibrium points must be unstable.

As a consequence of the fact that the solution operator of (2.1.1), (2.1.2) is compact for $t > 0$ and that the system is gradient, we have the following result.

THEOREM 2.1.2. *If system (2.1.1), (2.1.2) is point dissipative, then there exists a compact global attractor $\mathcal{A}$. If, in addition, the equilibrium points are hyperbolic, then*

$$\mathcal{A} = \cup_{\varphi \in E} W^u(\varphi).$$

2.2. Transversality of stable and unstable manifolds

Suppose that (2.1.1) has a compact global attractor. Since (2.1.1) is a gradient system, the flow on the attractor is Morse-Smale and thus structurally stable if the equilibrium points are hyperbolic and the stable and unstable manifolds are transversal (Hale, Magalhães and Oliva (1984)). The following rather surprising result is due to Henry (1985), Angenent (1986).

THEOREM 2.2.1. *If φ, ψ are hyperbolic equilibria of (2.1.1), then $W^u(\varphi)$ is transversal to $W^s(\psi)$ for every function f; that is, system (2.1.1) is Morse-Smale and thus structurally stable if and only if the equilibrium points are hyperbolic.*

REMARK 2.2.1. It is natural to enquire if the system (2.1.1) in several space variables also satisfies the property that stable and unstable manifolds of hyperbolic equilibrium points intersect transversally. Unfortunately, this is not true. For the unit ball in $\mathbb{R}^2$, POLÁČIK (1994) has shown that there is a function $f = f(x, u)$ for which transversality does not hold for the corresponding parabolic equation. Also, in this same class of functions $f(x, u)$, BRUNOVSKY and Poláčik (1995) have shown that generically (2.1.9) is a Morse-Smale system. It is not known if these same remarks hold if $f = f(u)$ does not depend upon x.

The proof of Theorem 2.2.1 involves several basic properties of the solutions of (2.1.1). For any function $v(t,x)$, $t \in \mathbb{R}$, $x \in (0,1)$, let $z(v(t,\cdot))$ be the number of zeros of $v(t,x)$ in $(0,1)$. An easy application of the maximum principle and the Jordan curve theorem yields the following result of Nickel (1962), Matano (1982).

Lemma 2.2.1. *If $a(t,x), b(t,x), c(t,x)$ are continuous functions for $t \in \mathbb{R}$, $x \in [0,1]$, and $v(t,x)$ is a solution of*

$$v_t = v_{xx} + a(t,x)v + b(t,x)v_x + c(t,x) \tag{2.2.1}$$

with the homogeneous boundary condition (2.1.2), then $z(v(t,\cdot))$ is a nonincreasing function of t.

This property together with the Sturm-Liouville theory puts strong restrictions upon equilibria if there is an orbit connecting them. The precise statement is

Lemma 2.2.2. *If φ, ψ are hyperbolic equilibria of (2.1.1), (2.1.2), and there is an orbit γ through a point η_0 with $\alpha(\eta_0) = \varphi$ and $\omega(\eta_0) = \psi$, then $\dim W^u(\varphi) > \dim W^u(\psi)$.*

The idea for the proof of Lemma 2.2.1 is the following. If $\eta(t)$ is a solution of (2.1.1) with $\alpha(\eta) = \varphi$, $\omega(\eta) = \psi$, then the linear variational equation around η is a linear equation of the form (2.2.1) with the coefficients converging exponentially as $t \to +\infty$ (resp. $t \to -\infty$) to the linear variational equation about ψ (resp. φ). As a consequence of this fact and the fact that no solution of (2.2.1) can approach zero faster than any exponential, it is to be expected that the zero number $z(\eta_t)$ of η_t should be at least the number of zeros of the eigenfunction corresponding to the largest eigenvalue of the linear variational equation about ψ which, by the Sturm-Liouville theory, is at least dim $W^u(\psi)$. By the same reasoning, for large negative t, $z(\eta_)$ should be no more that dim $W^u(\varphi) - 1$. Therefore, Lemma 2.2.1 implies that dim $W^u(\varphi) >$ dim $W^u(\psi)$.

To complete the proof of Theorem 2.2.1, one uses the characterization of $TW^s(\psi)$ as those functions which are orthogonal to the solutions of the adjoint linear equation

$$w_t = -w_{xx} - a(-t,x)w - b(-t,x)w_x - c(-t,x) \tag{2.2.2}$$

which approach $TW^u(\varphi)$ as $t \to \infty$. If $W^u(\varphi)$ is not transversal to $W^s(\psi)$, then there is a solution of (2.2.2) which approaches zero as $t \to \pm\infty$. One now proves that Lemma 2.2.2 is valid for (2.2.2) and deduces that $\dim W^u(\varphi) < \dim W^s(\psi)$ which is a contradiction.

Lemma 2.2.2 allows one to give an interesting Morse decomposition of the attractor $\mathcal{A}$ in the situation where all equilibrium points are hyperbolic. In fact, if we let

$$S_N = \{\varphi \in E : \dim W^u(\varphi) = N\},$$

then Theorem 2.1.1 implies that the α- and ω-limit set of any orbit must be an element of $\cup_N S_N$. Lemma 2.2.2 implies that N for the α-limit set is larger than N for the ω-limit set. This is equivalent to saying that the invariant sets $\{S_N, N = 0, 1, 2, \ldots\}$ form a *Morse decomposition* of the attractor A. To understand more about the nature of the flow on the attractor, it is necessary to understand the connecting orbits between these Morse sets.

In the special case of the Chafee-Infante problem,

$$u_t = \varepsilon^2 u_{xx} + u - u^3 \quad \text{in } (0, 1),$$

with homogeneous Neumann boundary conditions, Chafee and Infante (1974) discussed the bifurcation of equilibria as a function of $\varepsilon > 0$ and Henry (1985) has given a complete characterization of the attractor $\mathcal{A}_\varepsilon$ for each $\varepsilon > 0$. The equilibrium points ± 1 are stable for all ε and the equilibrium point 0 is unstable for all ε with its index $i(0) = i_\varepsilon(0)$ (the dimension of the unstable manifold) depending upon ε. Let φ_0^+ (resp. φ_0^-) denote the constant function 1 (resp. -1). At each point $\varepsilon_n = (n\pi)^{-1}$, $n = 1, 2, \ldots$, there is pitchfork bifurcation from the origin to equilibrium points φ_n^+, φ_n^- with $\varphi_n^+(x) = -\varphi^-(-x)$ having n zeros in (0,1). The equilibria $\varphi_n^\pm$ persist for $\varepsilon < \varepsilon_n$ and have index n. In the interval $(\varepsilon_{n+1}, \varepsilon_n)$, there are exactly $2n + 1$ equilibrium points, they all are hyperbolic and $i(0) = n$. For $\varepsilon \in (\varepsilon_1, \infty)$, $i(0) = 1$, and $\mathcal{A}_\varepsilon$ consists of the constant functions $\{-1, 0, 1\}$ together with $W^u(0)$. For $\varepsilon \in (\varepsilon_{n+1}, \varepsilon_n)$, $\mathcal{A}_\varepsilon = \operatorname{Cl} W^u(0)$ and, for any $1 \le k \le n$, there is an orbit connecting φ^k to φ_j^+ (resp. φ_j^-) (that is, an orbit whose α-limit set is φ_k and whose ω-limit set is φ_j (resp. φ_j^-)) for any $j < k$. The proof of these results uses the transversality of stable and unstable manifolds even at the bifurcation points. For a proof and a more detailed description of the attractor, see Henry (1985).

In a later section, we will give a constructive way to conclude that there is a connecting orbit between two hyperbolic equilibrium points φ, ψ.

Remark 2.2. *Slow motion* For the Chafee-Infante problem, we have described the flow on the attractor and have seen that there are only two stable equilibrium points ± 1 for all $\varepsilon > 0$ and that the dimension of the attractor approaches infinity as $\varepsilon \to 0$. If we start with initial data φ, then the solution $u(t)$ through φ will follow the following scenario. It will first come to a small neighborhood of the attractor and therefore to a small neighborhood of the unstable manifold of some equilibrium point. It will follow this unstable manifold until it comes to a small neighborhood of another unstable manifold and continue this process until it comes to a small neighborhood of one of the stable points ± 1. Numerical integration of the equation should follow this pattern. If ε is very small, one observes often that it is impossible to compute long enough to reach the stable points. More precisely, one observes that the solution reaches a state which has transition layers through zero and it remains in this state no matter how long the machine is allowed to run. The explanation for this is that the flow on the unstable manifolds of equilibrium points moves at a rate which is exponentially small as $\varepsilon \to 0$. A detailed discussion of this point involves a very delicate application of asymptotic analysis and invariant manifold theory and details can be found in CARR and PEGO (1989), (1990), FUSCO and HALE (1989), FUSCO (1990). More general results are in PINTO (1995). The ideas used in the analysis of these one dimensional problems also are useful in the discussion of slow motion of interfaces that occur in the Cahn-Hillard equation in a two dimensional domain (see ALIKAKOS, BATES and FUSCO (1991), ALIKAKOS and FUSCO (1992)).

It is interesting to study the transversality property for spatial discretization of the parabolic system. From this motivation, FUSCO and OLIVA (1988) showed that stable and unstable manifolds of hyperbolic equilibria are transversal for a very general class of ODE. We first state their general result and then relate it to spatial discretization.

DEFINITION 2.2.1. A bounded linear operator on $\mathbb{R}^n$ is said to a *positive Jacobi operator* if there is a set $e = \{e_i, 1 \le i \le n\}$ of basis vectors of $\mathbb{R}^n$ such that the matrix representation $A = (a_{ij})$ in this basis is a *positive Jacobi matrix*; that is, $a_{ij} = 0$ for $|i-j| > 1$ and $a_{ij} > 0$ if $|i-j| = 1$. We denote the set of all positive Jacobi matrices relative to the basis e by $\mathcal{J}(e)$.

The following result is due to FUSCO and OLIVA (1988).

THEOREM 2.2.2. *Suppose that $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$, U open, is a C^2-function and there is a basis $e = \{e_i, 1 \le i \le n\}$ in $\mathbb{R}^n$ such that, for all $z \in U$, the derivative $f'(z) \in \mathcal{J}(e)$. If z_-, z_+ are hyperbolic equilibrium points of the ODE*

$$\dot{z} = f(z) \tag{2.2.3}$$

with the property that there is an orbit connecting z_- to z_+, then $W^u(z_-)$ is transversal to $W^s(z_+)$.

An example of a function f with the property that $f'(z) \in \mathcal{J}(e)$ is the following:

$$\begin{aligned} f_1(z_1, z_2) &= a_1(z_1) + b_2 z_2, \\ f_j(z_{j-1}, z_j, z_{j+1}) &= a_j(z_j) + c_{j-1} z_{j-1} + b_{j+1} z_{j+1}, \quad 2 \le j \le n-1, \\ f_n(z_{n-1}, z_n) &= a_n(z_n) + c_{n-1} z_{n-1}, \end{aligned} \tag{2.2.4}$$

where the a_j, $1 \le j \le n$, are C^2-functions and the c_j, b_{j+1}, $1 \le j \le n-1$, are positive constants.

Let us consider the partial differential equation

$$u_t = u_{xx} + f(u) \quad \text{in } (0,1) \tag{2.2.5}$$

with say homogeneous Neumann boundary conditions. If we take an n point uniform discretization of $(0,1)$ and let $z_j = u(j/(n+1), t)$, then we obtain a system of the form (2.2.4) with $a_j(s) = f(s)$ for all s and all of the c_j, b_{j+1} equal to $(n+1)^2$. Therefore, Theorem 2.2.2 applies to this example and we have transversality of stable and unstable manifolds of hyperbolic equilibria.

If the function f in (2.2.5) depends upon x; that is, $f = f(x,u)$, then the transversality still holds for the discretized system since the constants in (2.2.4) remain the same and the functions $a_j(s) = f(j/(n+1), s)$. If we replace the diffusion term by $(b^2 u_x)_x$ with $b^2 > 0$ and continuous, then the same remark applies except now the positive constants off the diagonal may be different.

If we suppose that f in (2.2.5) depends upon u_x; that is, $f = f(u, u_x)$, and make the same type of spatial discretization, then the term u_x leads to some additional off diagonal terms (of course, on the tridiagonals). To keep the corresponding ODE so that the derivative belongs to $\mathcal{J}(e)$ with the same basis e, some limitations need to be imposed upon the size of the discretization. If we suppose that $|f_v(u,v)| \leq M$ for all u, v, then it is sufficient to consider step sizes corresponding to $n > M$ (Rocha (1994)).

In the proof of Theorem 2.2.2, a basic role is played by the number of sign changes in the components of the n-dimensional vector z, which is the analogue of the zero number of solutions of the PDE.

We say that a set $\mathcal{M}$ is an *inertial manifold* of (2.1.1), (2.1.2) if it contains the global attractor and is positively invariant under the flow. If we assume hyperbolicity of equilibria and point dissipativeness, then $\dim \mathcal{A} = \max_{\varphi \in E}\{\text{index}\,(\varphi)\}$.

Theorem 2.2.2. *If (2.1.1), (2.1.2) is point dissipative and the equilibria are hyperbolic, then there is a C^1 inertial manifold of minimal dimension and it is a graph over the linearized unstable manifold of maximal dimension.*

For the case where $f = f(u) = u - u^3$, Jolly (1989) proved the existence, Brunovsky (1989) proved the complete result for general $f(u)$ and Rocha (1991) allowed the dependence on the gradient.

2.3. Stable solutions with variable diffusion

In this section, we consider a scalar equation for wnich the reaction term involves no gradient terms and the diffusion coefficient depends upon the spatial variable. Our objective is to characterize those diffusion coefficients for which the only stable equilibrium solutions will be spatially independent.

We begin with the discussion of the first eigenvalue of the linear variational operator of an equilibrium point; namely, the elliptic nonlinear equation

$$(b^2(x) u_x)_x + f(u) = 0 \quad \text{in } (0,\,1), \tag{2.3.1}$$

where $b(x) > 0$ for all x and the function u is required to satisfy the Neumann boundary condition

$$u_x(0) = u_x(1) = 0. \tag{2.3.2}$$

The space of initial data is taken to be $H^1((0,\,1))$.

Suppose that v is a solution of (2.3.1), (2.3.2) and let $\lambda_1(v,\,b,\,f)$ be the first eigenvalue of the linear variational operator

$$L \equiv -\frac{\partial}{\partial x}(b^2\frac{\partial}{\partial x}) - f'(v)$$

on $H^1((0,\,1)$. Then $\lambda_1(v,\,b,\,f)$ is simple and there is an eigenfunction which is positive on $(0,\,1)$.

Equation (2.3.1) arises in a natural way as the equilibrium points of a scalar reaction diffusion

$$u_t = (b^2(x)u_x)_x + f(u) \quad \text{in } (0,\,1), \tag{2.3.3}$$

or the linearly damped hyperbolic equation

$$u_{tt} + \beta u_t = (b^2(x)u_x)_x + f(u) \quad \text{in } (0,\,1), \tag{2.3.4}$$

with homogeneous Neumann boundary conditions, where $\beta > 0$ is constant. It represents the equation for the equilibrium solutions. These systems are gradient with the energy functionals

$$\begin{aligned} V(\varphi) &= \int_0^1 \{\frac{1}{2}b^2[\varphi_x]^2 + F(\varphi)\}dx \\ V(\varphi,\psi) &= \int_0^1 \{\frac{1}{2}b^2[\varphi_x]^2 + \frac{1}{2}\psi^2 + F(\varphi)\}dx \end{aligned} \tag{2.3.5}$$

where $F(u) = -\int_0^u f(s)ds$.

The following result of Yanagida (1982) is a definitive statement on the role that the diffusion coefficient has on the first eigenvalue of the linear variational operator about an equilibrium. The first part of the theorem for the case where $b(x)$ is independent of x is due to Chafee (1975).

Theorem 2.3.1. *If b is a C^2-function, then the following statements hold:*

(i) If $b''(x) \le 0$ for all x and v is a nonconstant solution of (2.3.1), (2.3.2), then $\lambda_1(v,\,b,\,f) < 0$.

(ii) If there is an $x_0 \in (0,\,1)$ such that $b''(x_0) > 0$, then there is a C^2-function f such that (2.3.1), (2.3.2) has a nonconstant solution v such that $\lambda_1(v,\,b,\,f) > 0$.

The proof of the first part of the theorem uses the characterization of the first eigenvalue as the minimum of the functional $\mathcal{H}$ defined by

$$\mathcal{H}(\varphi) = \int_0^1 [b^2\varphi_x^2 - f'(v)\varphi^2]\,dx$$

over functions in $H^1(0,1)$ with L^2-norm equal to one.

To show that $\lambda_1 < 0$ if v is not a constant function, we show, following CHIPOT and HALE (1983), that $\mathcal{H}(bv_x) < 0$ if v is not a constant function. Let us compute the first term in the expression for $\mathcal{H}(bv_x)$.

$$\begin{aligned}\int_0^1 f'(v)(bv_x)^2 &= \int_0^1 f'(v)v_x b^2 v_x = \int_0^1 [f(v)]_x b^2 v_x = -\int_0^1 f(v)(b^2 v_x)_x \\ &= \int_0^1 [(b^2 v_x)_x]^2 = \int_0^1 [b(bv_x)_x + b_x(bv_x)]^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{H}(bv_x) &= -\int_0^1 (b_x b v_x)^2 + bb_x \cdot 2bv_x(bv_x)_x = -\int_0^1 (b_x b v_x)^2 + bb_x[(bv_x)^2]_x \\ &= \int_0^1 [-(b_x b v_x)^2 + (bb_x)_x(bv_x)^2] = \int_0^1 [-b_x^2 + (bb_x)_x](bv_x)^2 \\ &= \int_0^1 bb_{xx}(bv_x)^2 \leq 0.\end{aligned}$$

If $\lambda_1 \geq 0$, then the definition of λ_1 and the fact that $\mathcal{H}(bv_x) \leq 0$ imply that $\mathcal{H}(bv_x) = 0$ and thus $\lambda_1 = 0$. If φ_1 is an eigenfunction corresponding to λ_1, then the fact that λ_1 is simple and $\mathcal{H}(bv_x) = 0$ allow us to conclude that there is a constant c such that $bv_x = c\varphi_1$. If $c \neq 0$, then $v_x = 0$ at $x = 0, x = 1$, and $b \neq 0$ imply that we must have $\varphi_1 = 0$ at $x = 0$. Since $\varphi_{1x} = 0$ at $x = 0$, we would have that $\varphi_1(x) = 0$ for all x from uniqueness of the initial value problem in the ODE $Lu = 0$. This contradicts the fact that φ_1 is an eigenfunction. Thus, $c = 0$, $bv_x = 0$ and $v_x = 0$ which implies that v is a constant function.

The proof of the second part is more complicated. The idea is to construct a monotonically increasing function v which satisfies the boundary conditions and then construct a function f so that v is a solution of (2.3.1). This in done in such a way that the function f has a constant negative derivative $-B$ outside some interval $(-\alpha,\, \alpha)$ and then choose the constant B very large to ensure that v is stable. We refer to YANAGIDA (1982) for the details.

REMARK 2.3.1. The first part of Theorem 2.3.1 remains true for the system

$$\begin{aligned}-\Delta u &= f(u, \nabla u) \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 \quad \text{in } \partial\Omega\end{aligned}$$

provided that $\Omega \subset \mathbb{R}^N$ is convex and has a smooth boundary. More precisely, the first eigenvalue of the linear variational operator around a solution v of the equation is negative if v is nonconstant. In case f does not depend upon the gradient, this result is due to CASTEN and HOLLAND (1978), MATANO (1979) and the proof is very similar to the one given for the one dimensional case. The general case using gradient terms is due to HESS (1987) and the proof uses PDE arguments and especially the maximum principle.

For other boundary conditions, we have the following result.

THEOREM 2.3.2. *Consider the equation (2.3.1) with the mixed boundary conditions*

$$\alpha_0 u_x(0) + (1 - \alpha_0)u(0) = 0$$

$$\alpha_1 u_x(1) + (1 - \alpha_1)u(1) = 0$$

with α_0, α_1 being constants. If $b'' \leq 0$ and v is a nonconstant equilibrium solution such that $v_x = 0$ at two points in $[0, 1]$, *then* $\lambda_1(v, b, f) < 0$.

The proof is elementary. Let $v_x = 0$ at the points α, β with $\alpha < \beta$. Then v is a nonconstant solution of the Neumann problem on (α, β) and the first eigenvalue of the linear variational operator must be positive. By the variational characterization of the first eigenvalue of the linear variational operator for the solution of the mixed problem, it also must be positive.

THEOREM 2.3.3. *Consider either equation (2.3.3) or (2.3.4) with homogeneous Neumann boundary conditions (2.3.2). If b is a C^2-function, then the following statements hold:*

(i) If $b''(x) \leq 0$ for all x, then a stable equilibrium point must be spatially independent.
(ii) If there is an $x_0 \in (0, 1)$ such that $b''(x_0) > 0$, then there is a C^2-function f such that there is a stable equilibrium point which is spatially dependent.

For (2.3.3), this is an obvious consequence of Theorem 2.3.1. For (2.3.4), the linear variational equation about v has the eigenvalues μ satisfying the equation $\mu^2 + \beta\mu + \lambda_1(b, v, f) = 0$. The quadratic formula and Theorem 2.3.1 imply the result.

Of course, we can make a similar statement for the mixed boundary conditions concerning the instability of any equilibrium point v of (2.3.3), (2.3.4) such that $v_x = 0$ at two points in $[0, 1]$.

We emphasize that the instability of nonconstant equilibrium points holds for every function f. On the other hand, the existence of a stable nonconstant solution when $b''(x_0 > 0$ involves the construction of a special f. It is not to be expected that one can find a stable nonconstant solution for every f. In fact, suppose that $b(x) = 1$ for all x and the equilibria of (2.3.3), (2.3.4) are hyperbolic and there is a compact global attractor. Then there will be no stable nonconstant equilibria of (2.3.3), (2.3.4) if the diffusion coefficient is very close to one in the C^0-topology. This remark is a consequence of the fact that the equilibrium points are continuous with respect to b.

MATANO (1979) observed that it was possible to find a C^0-function b and a C^2-function f such that (2.3.3) has a stable nonconstant equilibrium. His proof of this fact used the parabolic structure of the equation. We sketch the idea and give more details in a more general context in Section 2.5. Suppose that $f(u) = u - u^3$. For given $0 < x_0 < x_1 < 1$, let $b(x)$ be very close to 1 on the intervals $(0,\, x_0)$ and $(x_1,\, 1)$ and very close to 0 on $(x_0,\, x_1)$. Since 1 and -1 are stable equilibria for (2.3.3), (2.3.2) with this b and f, we expect that there is a stable equilibrium which is close to 1 on $(0,\, x_0)$ and close to -1 on $(x_1,\, 1)$. This is proved by showing that there is an upper solution $\bar{u}$ and a lower solution $\underline{u}$ such that $\bar{u} < 0$ on $(x_1,\, 1)$ and $\underline{u} > 0$ on $(0,\, x_0)$. The region $[\underline{u},\, \bar{u}]$ is positively invariant under the flow. A general result of MATANO (1979) implies that such a region must contain a stable equilibrium. It is obviously nonconstant.

REMARK 2.3.2. As a consequence of Remark 2.3.1, equation (2.1.1) on a bounded convex domain $\Omega \subset \mathbb{R}^n$ with Neumann boundary conditions has the property that the only stable equilibrium points must be spatially independent. If the domain is not convex, then MATANO (1979), using the theory outlined in Section 2.5, has given an example of a domain Ω and a nonlinear function $f(u)$ for which there is a stable nonconstant equilibrium. The idea is similar to the construction above taking a domain which is dumbbell shape with a thin channel and a function $f(u)$ which is bistable. Space prevents a detailed discussion of problems of this type. The reader should consult CIUPERCA (1996), HALE and VEGAS (1984), MORITA (1990), S. OLIVA (1995) and the references therein as well as Mathemataical Reviews.

In the applications, the equation for the equilibrium solutions of the evolutionary PDE may have a form which is different from but equivalent to (2.3.1), (2.3.2). For example, let us consider the equation

$$u_t - \Delta u = f(u) \quad \text{on } \Omega_\varepsilon \tag{2.3.6}$$

with homogeneous Neumann boundary conditions, where $\Omega_\varepsilon = (x, y) : x = \varepsilon g(x), x \in (0, 1)\}$, $\varepsilon > 0$ is a small real parameter, $g > 0$ is a C^3-function and f is a C^2-function. HALE and RAUGEL (1992) have shown that the dynamics on the attractor A_ε for ε small is captured by the dynamics on the attractor of the limit equation on $(0, 1)$:

$$u_t - \frac{1}{g}(g(x)u_x)_x = f(u) \quad \text{in } (0,\, 1) \tag{2.3.7}$$

with homogeneous Neumann boundary conditions. Without loss of generality, we can suppose that $\int_0^1 g(s)\, ds = 1$. If u is any equilibrium point of (2.3.7) and if we let $y = \int_0^x g(s)\, ds$, $u(x) = v(y)$ and $b(y) = g(x(y))$, then $u_x = bv_y$ and the function v satisfies the equation (2.3.1). This transformation also preserves the boundary condition (2.3.2). After a few calculations, we deduce that

$$
\begin{aligned}
b''(y) &= (g(x(y)))^2 H(x(y)), \\
H &= -\frac{(\log g)'g' + g'' = -g')^2}{g} + g''.
\end{aligned}
\tag{2.3.8}
$$

Applying Theorem 2.3.1 to (2.3.7) yields interesting information about the effect of domain shape on dynamics. In fact, one can prove the following unpublished result of Hale and Raugel.

Theorem 2.3.4. *If H is defined as in (2.3.8), then the following statements hold:*

(i) If $H(x) \leq 0, 0 \leq x \leq 1$, and the zeros of f are simple, then there is an $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, any nonconstant equilibrium solution of (2.3.7) is unstable.

(ii) If there is an $x_0 \in (0, 1)$ such that $H(x_0) > 0$, then there is a C^2-function f and an $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, (3.2.7) has a stable nonconstant solution.

It also is possible to make other transformations on (2.3.7) to obtain interesting forms for the equations. For example, let us assume (without loss of generality) that $\int_0^1 g(s)^{-1}\, ds = 1$. If we let the function $h(y)$ be determined from the differential equation $h' = g(h)$, $h(0) = 0$, and let $x = h(y)$, $u(x) = v(y)$, $s(y) = g(h(y))$, $d(y) = s^2(y)$, then

$$
v_{yy} + d(y)f(v) = 0 \quad \text{in } (0,\, 1). \tag{2.3.9}
$$

The boundary conditions (2.3.2) also are preserved. If we let v be a solution of (2.3.9), (2.3.2) and let $\tilde{\lambda}_1(v,\, d,\, f)$ be the first eigenvalue of $-\partial^2/\partial x^2 - df'(v)$ on $H^1((0,\, 1))$, then, since $d = s^2$, an application of Theorem 2.3.1 implies that $\tilde{\lambda}_1(v,\, d,\, f) < 0$ for any nonconstant v provided that $-2s'(y) + [h(y)]^{-1}s(y)s''(y) \leq 0$ for all y. If this condition is not satisfied at some point, then then there is an f such that $\tilde{\lambda}_1(v,\, s,\, f) > 0$ for some nonconstant solution of (2.3.9). This implies that there exists a stable nonconstant solution of the parabolic equation

$$
v_t - v_{xx} = d(x)f(v) \quad \text{in } (0,\, 1), \tag{2.3.10}
$$

as well as the linearly damped hyperbolic equation

$$
v_{tt} + \beta v_t - v_{xx} = d(x)f(v) \quad \text{in } (0,\, 1), \tag{2.3.11}
$$

with boundary condition (2.3.2).

We remark that the above transformation applied to (2.3.7) preserves homogeneous Dirichlet boundary conditions, but mixed boundary conditions are not preserved unless $g(0) = g(1) = 1$.

When we arrive at (2.3.11) from (2.3.7) through special transformations, the function d is positive. In some applications, the equation (2.3.10) occurs with d not of fixed sign. We discuss the existence of stable nonconstant equilibria for such equations in the next section.

Let us now suppose that the diffusion coefficient in (2.3.1) depends upon a parameter $\mu \in \mathbb{R}$; that is, $b = b_\mu$. It is of interest to determine how variations in the diffusion coefficient can lead through bifurcations to a nonconstant solution v_μ of (2.3.1), (2.3.2) with $\lambda_1(v_\mu,\, b_\mu,\, f) > 0$ starting for example with $b_0 = 1$. It is clear that, if c is a constant solution with $\lambda_1(c,\, b_0,\, f) > 0$, then $\lambda_1(c,\, b_\mu,\, f) > 0$ for all μ. This is a consequence of the easily observed fact that $\lambda_1(c, b_0, f) > 0$ implies that $f'(c) < 0$. Therefore, any primary bifurcation from a constant solution of (2.3.1), (2.3.2) must be from a solution v with $\lambda_1(v_\mu,\, b_\mu,\, f) < 0$ and thus is unstable. This implies that the additional stable solutions must arise either from secondary bifurcations on from saddle node bifurcations. With enough symmetry on b and f, one can eliminate the consideration of saddle node bifurcations.

Assuming that f in (2.3.1) is odd, dissipative and has three simple zeros $\pm 1, 0$, $f'(u) < f'(0)u, u \in (0, 1]$, and that $b = b_\mu$ is even about $x = 1/2$, depends continuously upon a parameter μ, b_μ is close to the step function b_0 which is 1 on $[0, 1/2 - \beta] \cup [1/2 + \beta, 1]$ ($\beta < 1/2$) and γ on $(1/2 - \beta, 1/2 + \beta)$ with γ sufficiently small, Fusco and Hale (1985) have discussed the manner in which stable nonconstant equilibria may arise through secondary bifurcations as stated in the following result.

Theorem 2.3.5. *Under the above conditions on b_μ, f, if φ_μ is an equilibrium point of (2.3.1), (2.3.2) which is zero for $\mu = 0$ and has exactly k zeros for $\mu \in (0, 1)$, depends continuously on μ, and φ_1 is stable, then there exist numbers $0 < \mu_1 < \mu_2 < \cdots < \mu_k < 1$ such that each $\mu_i, i = 1, 2, \ldots, k$, is a bifurcation point.*

Theorem 2.3.5 says that going through k secondary bifurcations is a necessary condition in order that an equilibrium point with k zeros becomes stable after it bifurcates from the zero solution. Fusco and Hale (1985) also show that if, as μ goes from 0 to 1, an equilibrium point φ_μ experiences exactly k bifurcations at $0 < \mu_1 < \cdots < \mu_k < 1$, each one of which is generic in the sense that, at any μ_i, two new solutions bifurcating from μ_i appear, then u_1 is stable. This observation shows that in a certain sense the converse of Theorem 2.3.5 is true.

Hale and Rocha (1985) have discussed possible bifurcation diagrams in more detail for the particular case where $f(u) = \delta(u - u^3)$, $\delta > 0$ constant, and

$$b(x) = \begin{cases} 1, & \text{if } 0 \le x \le \gamma, \\ \alpha, & \text{if } \gamma < x \le 1, \end{cases} \tag{2.3.12}$$

with $0 < \alpha, \gamma < 1$. In this case, they show that there are no stable nonconstant equilibrium points for any δ. If we were to replace b by a continuous function, then Yangida's Theorem 2.3.3 would imply that there is an f such that there are stable nonconstant equilibrium points.

If we require that $0 < \gamma < 1/2$ and define b as above on $(0, 1/2)$ and to be even about $1/2$, Hale and Rocha (1985) show that there are exactly two stable nonconstant equilibrium solutions which bifurcate from zero and become stable through a secondary bifurcation. For the bifurcation parameter, one can use $\alpha^{-1}\delta^{1/2}$.

For a domain Ω which is the unit ball, Nascimento (1983), (1990), has a partial extension of the Theorem 2.3.3. Assuming that the equation has the form

$$u_t = \nabla \cdot (a(|x|)\nabla u) + f(u)$$

with homogeneous Neumann boundary conditions on the unit ball in $\mathbb{R}^N$, he shows that a stable equilibrium must be radially symmetric. Furthermore, he gives good estimates on the form of a so that the only stable equilibria are constants for every function f. Finally, he constructs an $(a,\, f)$ such that there exists a stable nonconstant radially symmetric solution.

Remark 2.3.2. Subsidiary conditions of a nonlocal character will automatically change the spatial structure of the solutions v of (2.3.1), (2.3.2) which have the property that $\lambda_1(v,\, b,\, f) > 0$. For example, if we require in (2.3.1), (2.3.2) that

$$\int_0^1 u(t,\, x)\, dx = c \neq 0, \tag{2.3.13}$$

where c is a constant not equal to any of the zeros of f, then each solution must have spatial structure. For $b = 1$, it is shown in Carr, Gurtin and Slemrod (1984) that any solution v with $\lambda_1(v,\, b,\, f) > 0$ must be a monotone nonconstant function if f is the bistable function $f(u) = u(u-a)(1-u)$, $0 < a < 1/2$. This implies that the only stable solutions of (2.3.3), (2.3.2), with $b = 1$, $f(u) = u(u-a)(1-u)$, $0 < a < 1/2$, and satisfying (2.3.13) with $c \notin \{0, a, 1\}$ are monotone and nonconstant. It would be interesting to determine the necessary and sufficient conditions on b so that this same conclusion is valid for (2.3.3).

For more general results for the equation

$$u_t = u_{xx} + f(u) - \alpha \int_0^1 f(u(x, \cdot))dx,$$

where α is a constant ($\alpha = 1$ corresponds to the case just mentioned), see Freitas (1993).

For the strongly damped nonlinear wave equation

$$u_{tt} + \beta u_t - c_2 u_{xx} - c_1 u_{xxt} = f(u) \quad x \in (0, 1)$$

with $u_x = 0$ at $x = 0, 1$, and c_1, c_2 positive constants, it is easy to check that nonconstant equilibrium solutions are unstable. What happens when c_1, c_2 are spatially dependent or when the spatial derivative terms are given as $-(c_1 u_x)_x - (c_2 u_{xt})_x$?

2.4. Primary bifurcation to stable nonconstant equilibria

In this section, we consider the equation

$$u_t - u_{xx} = \lambda s(x) f(u), \quad 0 < x < 1 \tag{2.4.1}$$

with the Neumann boundary conditions

$$u_x(0,t) = 0 = u_x(1,t) \tag{2.4.2}$$

and $\lambda > 0$ a real scalar parameter.

We assume that s is continuous and f is C^2. If $s(x) > 0$ for all $x \in [0,1]$, then the transformations used in the previous section show that there can be no primary bifurcation from an equilibrium point corresponding to a zero of f. We consider in this section the situation when $s(x)$ can change sign in $(0,1)$ and is positive on some subinterval of $(0,1)$. We also assume that

$$f(0) = 0 = f(1), \quad f'(0) > 0, \quad f'(1) < 0, \quad f''(u) < 0,\ 0 < u < 1. \tag{2.4.3}$$

Equation (2.4.1) with s changing sign is motivated by a selection migration model introduced by Fisher in population genetics. The existence of stable nonconstant solutions under some conditions on s has been shown by several authors, including Fleming (1975), Hoppenstadt (1975), Fife and Peletier (1981), Henry (1981), Brown, Lin and Tertikas (1989), Brown and Hess (1990). Our presentation follows Henry (1981). In the model, u represents the density of the population, the constant λ is essentially the ratio of the intensity of selection to the migration rate, $s(x)$ is the local relative selective advantage if $s(x) > 0$ and disadvantage if $s(x) < 0$.

From the maximum principle, it is easy to see that the complete metric space X of $H^1(0,1)$ consisting of all functions φ with values in $[0,1]$ is positively invariant under the flow defined by (2.4.1), (2.4.2). Therefore, we can consider (2.4.1), (2.4.2) as defining a dynamical system on X and having a compact global attractor $\mathcal{A}$ in X. Also, the system is gradient and we let E_λ denote the set of equilibrium points in X. The points $u = 0$ and $u = 1$ are trivial equilibrium points.

We want to study the dynamics on the attractor $\mathcal{A}$. The average $\bar{s} = \int_0^1 s(x)dx$ will play a fundamental role.

Lemma 2.4.1. *If $\bar{s} \neq 0$, then there is a $\lambda_0 > 0$ such that $E_\lambda = \{0,1\}$ if $0 < \lambda \leq \lambda_0$; that is, there are no nontrivial equilibrium points.*

The proof is easily supplied by using the method of Lyapunov-Schmidt. In fact, if $P : X \to X$ is the projection which takes $\varphi \in X$ to its average $\bar{\varphi} = \int_0^1 \varphi$, if $\varphi \in E_\lambda$, $\varphi = \bar{\varphi} + \psi$, then it is easy to see that the equation for ψ:

$$\psi_{xx} = \lambda(I - P)sf(\bar{\varphi} + \psi),$$

with Neumann boundary conditions, has a unique solution $\psi(\lambda, \bar{\varphi})$ which is C^1 in λ, $\bar{\varphi}$ and $\psi(0, \bar{\varphi}) = 0$. Therefore, $\bar{\varphi} + \psi(\lambda, \bar{\varphi})$ is an equilibrium point of

(2.4.1), (2.4.2) if and only if $\int_0^1 s(x)f(\bar{\varphi}+\psi(\lambda,\bar{\varphi}(x))dx = 0$. For $\lambda = 0$, this implies that $f(\bar{\varphi}) = 0$ since $\bar{s} \neq 0$. The Implicit Function Theorem implies that 0 and 1 are isolated equilibria for λ small and the proof is complete.

The following result is due to Henry (1981).

Theorem 2.4.1. *If $\bar{s} < 0$ and $s > 0$ on some interval, then the attractor for (2.4.1), (2.4.2) on X is a C^1-curve which, for $\lambda < \lambda_0$, consists of the two equilibria $0, 1$ and an orbit (the unstable manifold of 1) connecting 1 to 0. For $\lambda > \lambda_0$, the attractor consists of the equilibrium points $0, 1$ and a spatially dependent equilibrium point φ_λ together with the unstable manifolds of 0 and 1.*

In the proof, it is shown that the point λ_0 is a point of primary bifurcation from the equilibrium point 0. The detailed proof consists in first showing that, for the largest eigenvalue $\zeta(\lambda)$ of $\partial^2 + \lambda s f'(0)$, there is a unique $\lambda_0 > 0$ such that $\zeta(\lambda_0) = 0$, and the corresponding eigenfunction is positive. It is then easily shown that λ_0 is a primary supercritical bifurcation from 0 by using the method of Lyapunov-Schmidt. It is then shown that there can be no bifurcation from a nontrivial equilibrium point. From these facts, one deduces the conclusions in the theorem.

Remark 4.2. The assertions above hold also for equation (2.4.1) with the Laplacian on a bounded smooth domain $\Omega \subset \mathbb{R}^N$ (see Henry (1981)). For other related results without assuming the concavity of f, see Brown, Lin and Tertikas (1989), Brown and Hess (1990).

2.5. Stable equilibrium points

In the first subsection of this section, we give a general method of Matano (1979) for showing the existence of a stable equilibrium point for a parabolic partial differential equation. A particular consequence of the method is that any positively invariant set must contain a stable equilibrium. More abstract results and ones that are even valid for fixed points of maps are contained in the literature (see Matano (1984), Hirsch (1988), Dancer and Hess (1991), Smith (1995)). In the second subsection, we present a general method for determining the index of an hyperbolic equlibrium point in one space dimension.

2.5.1. Existence of stable equilibrium points Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and consider the general parabolic equation

$$u_t = Lu + f(x,u) \quad \text{in } \Omega \tag{2.5.1}$$

with the boundary conditions

$$\alpha(x)u + [1-\alpha(x)]\frac{\partial u}{\partial \nu} = \beta(x) \quad \text{in } \partial\Omega, \tag{2.5.2}$$

The functions α, β are C^2, $0 \leq \alpha \leq 1$, the operator L is a second order uniformly elliptic operator of the form

$$(Lu)(x) = \S_{i,j=1}^n \frac{\partial}{\partial x_i}[a_{ij}(x)\frac{\partial}{\partial x_j}u(x)]$$

with coefficients which are C^3 and $\partial/\partial\nu$ is the unit outward conormal derivative. The function $f(x,u)$ is in $C_x^1 \cap C_u^3$.

It is possible to define a solution of (2.5.1), (2.5.2) on several different spaces. For the purposes of this section, we take the initial data to be in the space $C(\bar{\Omega})$. For any $\varphi \in C(\bar{\Omega})$, it is possible to prove that there is a solution $T(t)\varphi$ of the system defined on the interval $[0, t_{\max}(\varphi))$ and $T(0)\varphi = \varphi$. Furthermore, $T(t)\varphi \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ for $t \in (0, t_{\max}(\varphi))$.

For simplicity in the discussion, we are going to assume that all solutions are defined on $[0,\infty)$. The family of maps $T(t), t \geq 0$, forms a C^1-semigroup on $C(\bar{\Omega})$. Also, $T(t)$ is a compact map for $t > 0$ and, in fact, if B is a bounded set in $C(\bar{\Omega})$, then $T(t)$ is in a bounded set in $C^1(\bar{\Omega}) \cap C^2(\Omega)$ for any $t > 0$ (see, for example, Matano (1979)).

Associated with (2.5.1), (2.5.2) is the energy functional

$$\begin{aligned} V(\varphi) = \quad & \int_\Omega [\frac{1}{2}\S_{i,j=1}^n a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j} + F(x,\varphi)]dx \\ & + \int_{\partial\Omega} a[\frac{\alpha}{2}\varphi^2 + \frac{1-\alpha}{2}\frac{\partial\varphi}{\partial\nu})^2 - \beta\varphi]ds, \end{aligned}$$

where

$$a(s) = [\S_i\{\S_j a_{ij}(s)n_j(s)\}^2]^{1/2}$$

and $(n_1, \ldots, n_n)$ is the outer normal of $\partial\Omega$ and $F(x,u) = -\int_0^u f$. It is not difficult to show that, if $u(t) = T(t)\varphi$ is a solution of (2.5.1), (2.5.2), then $\dot{V}(u(t)) = -\int_\Omega [u_t]^2 dx \leq 0$. If we assume also that $F(u) \to \infty$ as $|u| \to \infty$, then this latter inequality shows that the system is gradient and that $\omega(\varphi)$ belongs to the set E of equilibrium points; that is, the set of the solutons of the equation

$$L\varphi + f(x,\varphi) = 0 \tag{2.5.3}$$

which satisfy the boundary conditions (2.5.2).

To proceed further, we need some notation. Let

$$\begin{aligned} C^+(\varphi) \quad &= \{\psi \in C(\bar{\Omega}) : \psi \geq \varphi, \psi \not\equiv \varphi\}, \\ C^-(\varphi) \quad &= \{\psi \in C(\bar{\Omega}) : \psi \leq \varphi, \psi \not\equiv \varphi\}, \\ E^+(\varphi) \quad &= E \cap C^+(\varphi), \\ E^-(\varphi) \quad &= E \cap C^-(\varphi), \end{aligned}$$

and E is the set of equilibria. On these sets, we introduce the partial order $\psi_1 \geq \psi_2$ if $\psi_1(x) \geq \psi_2(x)$ for $x \in \bar{\Omega}$.

We say that an equilibrium point φ is *unstable upward* if there exists an $\varepsilon_0 > 0$ such that, for any $\delta > 0$, there exists a function $\psi \in C(\bar{\Omega})$ which satisfies $\varphi(x) \leq \psi(x) \leq \varphi(x) + \delta$ for all $x \in \Omega$ and there exist $x_0 \in \Omega$ and $t_0 > 0$ such that

$$T(t_0)\psi(x_0) \geq \varphi(x_0) + \varepsilon_0. \tag{2.5.4}$$

The point φ is said to be *strongly unstable upward* if there exists an $\varepsilon_0 > 0$ such that, for any $\psi \in C^+(v)$, there is an $x_0 \in \Omega$ and $t_0 > 0$ such that (2.5.4) holds. *Downward instability* is defined in a similar way by reversing the inequalities.

One of the main results is the following

THEOREM 2.5.1. *If $\varphi \in E$ is strongly unstable upward and $E^+(\varphi) \neq \emptyset$, then there exists a minimum $\varphi^+ \in E^+(\varphi)$; that is, $\varphi^+ \leq \psi$ for all $\psi \in E^+(\varphi)$. Moreover, for any $\psi \in C^+(\varphi) \cap C^-(\varphi^+)$, we have $\omega(\psi) = \{\varphi^+\}$.*

The main ingredients of the proof of Theorem 2.5.1 are the following lemmas.

LEMMA 2.5.1. *If $\varphi \in E$ is strongly unstable upward, then φ is isolated in $E^+(\varphi)$ in the topology of $C(\bar{\Omega})$ and $\omega(\psi) \in E^+(\varphi)$ for any $\psi \in C^+(\varphi)$.*

LEMMA 2.5.2. *If $\varphi \in E$ is strongly unstable upward and $E^+(\varphi) \neq \emptyset$, then, for any $\varphi_1, \varphi_2 \in E^+(\varphi)$, there exists the greatest lower bound of $\{\varphi_1, \varphi_2\}$ in $E^+(\varphi)$.*

To state more specific results, we need additional hypotheses on the boundary conditions.

(**H.1**) $\alpha \equiv 1$; that is, Dirichlet conditions.
(**H.2**) $1 - \alpha$ never vanishes on $\partial\Omega$; that is, Neumann or Robin conditions.

THEOREM 2.5.2. *If either (**H.1**) or (**H.2**) holds and $\varphi \in E$ is unstable upward (resp. downward), then it is strongly unstable upward (resp. downward).*

COROLLARY 5.1. *If either (**H.1**) or (**H.2**) holds and $\varphi \in E$, then φ is stable from above (that is, not unstable upward) if either of the following conditions are satisfied:*

(1) There is a $\psi \in C^+(\varphi)$ such that $\omega(\psi) = \varphi$.
(2) $\inf_{v \in E^+(\varphi)} v = \varphi$.

The proof of Theorem 2.5.2 is a consequence of the smoothness properties of $T(t)\psi$ in t, ψ, the strong maximum principle and the variation of constants formula.

Of course, analogous assertions hold for stability from below.

We give now sufficient conditions for the existence of a stable solution. A closed set $Y \subset C^1(\bar{\Omega}) \cap C^2(\Omega)$ has *Property (S)* if there exists a family of sets $\{Y_\alpha\}_{\alpha \in A}$ in $C^1(\bar{\Omega}) \cap C^2(\Omega)$ which satisfies the following proporties:

(1) $\cap_{\alpha \in A} Y_\alpha = Y$.
(2) For any $\alpha_1, \alpha_2 \in A$, there exists an $\alpha_3 \in A$ such that $Y_{\alpha_3} \subset Y_{\alpha_1} \cap Y_{\alpha_2}$.
(3) Each Y_α is closed in $C^1(\bar{\Omega}) \cap C^2(\Omega)$ and bounded in $C(\Omega)$.
(4) Given any $w \in Y$, any $\alpha \in A$ and any $\delta > 0$, there exist $w_1, w_2 \in C(\bar{\Omega})$ and a $t_0 > 0$ such that

$$w - \delta \le w_1 \le w \le w_2 \le w + \delta, \quad w_1, w_2 \not\equiv w,$$

and $T(t) w_i \in Y_\alpha, i = 1, 2$, for $t \in [t_0, \infty)$.

Remark 2.5.1. A set Y is said to be *stable* if for any neighborhood $U \subset C(\bar{\Omega})$ of Y there is a neighborhood $V \subset C(\bar{\Omega})$ of Y such that $T(t)V \subset U$ for $t \ge 0$. If Y is a stable set, then it has Property (S). In fact, Y is stable if and only if, for any neighborhood $U \subset C(\bar{\Omega})$ of Y, there is a neighborhood $U' \subset C(\bar{\Omega})$ of Y such that $T(t)U' \subset U'$ for $t \ge 0$. Therefore, we may take a sequence of neighborhoods $U_\alpha \subset C(\bar{\Omega})$ with $U_{\alpha_1} \subset U_{\alpha_2}$ if $\alpha_1 \ge \alpha_2$ and choose $Y_\alpha = \bar{U}_\alpha \cap (C^1(\bar{\Omega}) \cap C^2(\Omega))$.

Remark 2.5.2. Suppose that u_- (resp. u^+) is a subsolution (resp. supersolution) of (2.5.1), (2.5.2). Then the order interval $[u_-, u^+]$ has Property (S).

Theorem 2.5.3. *If either (**H.1**) or (**H.2**) holds and and $Y \neq \emptyset$ satisfies Property (S), then Y contains at least one stable equilibrium point.*

We outline the details of the proof. There is a compact attractor $\mathcal{A} \subset Y$ and therefore an element $\varphi_0 \in E \cap \mathcal{A}$. If Y contains no stable point, then either φ_0 is unstable upward of unstable downward. Without loss of generality, suppose that φ_0 is unstable upward. From Theorem 2.5.2, φ_0 is strongly unstable upward.

Let Z be the set of all elements of $E \cap Y$ which are unstable upward. Since Z can be considered as a partially ordered subset of Y, we can let M be the set of all the well ordered subsets of Z. We can regard M as a partially ordered set with respect to the order relation $W_1 \le W_2$ if W_1 coincides with a segment of W_2. The set M is inductively ordered and not empty since it contains φ_0. By Zorn's Lemma, there is a maximal element W of M. There are now two cases to consider.

Case 1. Suppose that W has a largest element ψ. It is first observed that the boundedness of each Y_α implies that the set $E^+(\psi)$ will have a minimum ψ^+. Furthermore, it will be stable from below. Since there are no stable points in Y, we must have ψ^+ unstable from above. Therefore, ψ^+ belongs to Z and $W \cup \{\psi^+\}$ belongs to M. This contradicts the fact that M is maximal.

Case 2. Suppose that W does not have a largest element. If $\hat{\psi}(x) = \sup_{\psi \in W} \psi(x)$, then Y bounded in $C(\bar{\Omega})$ implies that $\hat{\psi}$ is an accumulation point of W in the topology of $C^1(\bar{\Omega}) \cap C^2(\Omega)$ and thus $\hat{\psi} \in E$. The point $\hat{\psi}$ is stable from below and we have $\hat{\psi} \in Z$ since there are no stable points in Y. As before, this is a contradiction and the theorem is proved.

The following results are consequences of Theorem 2.5.1 and Theorem 2.5.3.

THEOREM 2.5.4. *If either (**H.1**) or (**H.2**) holds and φ_1, φ_2 are distinct elements of E, $\varphi_1 \le \varphi_2$ on $\bar{\Omega}$ with φ_1 unstable upward and φ_2 unstable downward, then*

(1) there is a minimal $\varphi_3 \in E$ and a maximal $\varphi_4 \in E$ satisfying $\varphi_1 \le \varphi_3 \le \varphi \le \varphi_4 \le \varphi_2$, $\varphi_1 \not\equiv \varphi_3$, $\varphi_4 \not\equiv \varphi_2$, for any $\varphi \in E \cap [\varphi_1, \varphi_2]$, $\varphi \not\equiv \varphi_1, \varphi_2$;
(2) there exists at least one stable $\varphi \in [\varphi_1, \varphi_2]$.

THEOREM 2.5.5. *If either (**H.1**) or (**H.2**) holds and $\varphi_1, \varphi_2 \in E$ are distinct, $\varphi_1 \le \varphi_2$ with φ_1 being stable from above and φ_2 being stable from below, then there is at least one other $\varphi \in E \cap [\varphi_1, \varphi_2]$.*

Theorem 2.5.3 together with Remark 2.5.2 may be used to verify the assertion in Remark 2.3.2 on the existence of a stable nonconstant equilibrium solution on a dumbbell shaped domain (see MATANO (1979)). Another application is given in Section 2.6.

2.5.2. The index of equilibria in one space dimension In this section, we consider the elliptic equation

$$(b^2 u_x)_x + f(x, u) = 0, \qquad 0 < x < 1\,, \tag{2.5.5}$$

with the Neumann boundary conditions

$$u_x(0) = 0 = u_x(1)\,, \tag{2.5.6}$$

where $b > 0$ is continuous and $f(x, u)$ is continuous in x, u with continuous first and second derivatives in u and

$$f(x, 0) = 0\,. \tag{2.5.7}$$

Let v be a solution of (2.5.5), (2.5.6) and let $i(v)$, the *index of* v, be the number of negative eigenvalues of the operator

$$\frac{\partial}{\partial x}(b^2 \frac{\partial}{\partial x}) + f_u(x, v(x))$$

with boundary conditions (2.5.6). Our objective in this section is to give a method for determining $i(v)$ which has been used by FUSCO and HALE (1985), HALE and ROCHA (1985) and ROCHA (1985). Similar methods have been used by JONES (1984) in the study of the stability of traveling waves.

Along with (2.5.5) and (2.5.6), we consider the initail value problem

$$
\begin{aligned}
b^2 u_x &= v, \quad v_x = -f(x,u), \quad 0 < x < 1, \\
u(0) &= u_0, \quad v(0) = 0,
\end{aligned}
\tag{2.5.8}
$$

where $u_0 \in \mathbb{R}$. Let $((u(x,u_0), v(x,u_0))$ be the solution of (2.5.8) and suppose it is defined on $[0,1]$ for each u_0. The set E of equilibrium solutions of (2.5.5), (2.5.6) coincides with those solutions of (2.5.8) for which $v(1,u_0) = 0$. If we let $\S \subset \mathbb{R}^3$ be defined by $\S = \{(u(x,u_0), v(x,u_0), x) : 0 \le x \le 1, u_0 \in \mathbb{R}\}$ and let $\S_x = \{(u,v) : (u,v,x) \in \S\}$ be the cross-section of $\S$ at x, then $u(\cdot,u_0) \in E$ if and only if

$$(u(1,u_0), v(1,u_0)) \subset \S_1 \cap \{(u,0,1) \in \mathbb{R}^3\}.$$

We need also the linear variational equation with respect to u_0:

$$
\begin{aligned}
b^2 \eta_x &= \mu, \quad \mu_x = -f_u(x,u)\eta, \quad 0 < x < 1, \\
\eta(0) &= 1, \quad \mu(0) = 0.
\end{aligned}
\tag{2.5.9}
$$

For the study of the index of an equilibrium point, an important role is played by the angle t (measured clockwise) that the tangent vector to the curve $\S_1$ makes with the u-axis. The tangent vector to $\S_x$ is given by

$$(\partial u(x,u_0)/\partial u_0, \partial v(x,u_0)/\partial u_0)$$

and satisfies the linear variational equation. Therefore, it is natural to introduce the polar coordinates $\eta = \rho \cos t, \mu = -\rho \sin t$ in (2.5.9) to obtain the following equation for the angle $t(x,u_0)$ that the tangent vector to $\S_x$ makes with the u-axis:

$$t_x = \frac{\sin^2 t}{b^2} + f_u(x, u(x,u_0)) \cos^2 t, \quad t(0,u_0) = 0. \tag{2.5.10}$$

This transformation is valid since we have assumed that $f(x,0) = 0$ for all x.

For any real number $y \in (-1,\infty)$, we let $[y]$ be the integer part of y if $y \ge 0$ and be -1 if $y < 0$.

THEOREM 2.5.6. *Let $\theta(u_0) = t(1,u_0)$, where t is the solution of (2.5.10). Then*

(1) $\theta : \mathbb{R} \to (-\pi/2, \infty)$.
(2) *An equilibrium point $u = u(\cdot,u_0)$ is hyperbolic if and only if $\theta(u_0) \ne k\pi$ for any integer $k \ge 0$; that is, the curve $\S_1$ is transversal to $\{(u,0,1) \in \mathbb{R}^3\}$ at u_0.*
(3) *If $u(\cdot,u_0)$ is an hyperbolic equilibrium and $W^u(u_0)$ is the unstable manifold of u, then* $\dim W^u(u_0) = 1 + [\theta(u_0)/\pi]$.

For the proof, note that t_x is positive for $t = (k - 1/2)\pi$, where $k \geq 0$ is an integer. Therefore, $t(0, u_0) = 0$ implies that t can never attain the value $-\pi/2$ and the range of θ must be in $(\pi/2, \infty)$.

If u is an equilibrium point, then the corresponding eigenvalue problem is

$$(b^2 w_x)_x + f'(x, u)w = \lambda w, \quad x \in (0, 1),$$

with the boundary conditions $w_x = 0$ at $x = 0, 1$. If we let

$$w = \eta \cos \xi, \quad b^2 w_x = -\eta \sin \xi,$$

then λ is an eigenvalue if and only if ξ satisfies the equation

$$\begin{aligned} \xi_x &= \tfrac{\sin^2 \xi}{b^2} + [f_u(x, u(x, u_0)) - \lambda] \cos^2 \xi, \quad x \in (0, 1), \\ \xi(0) &= 0, \quad \xi(1) = k\pi \text{ for some integer } k \geq 0. \end{aligned} \tag{2.5.11}$$

For $\lambda = 0$, this is the equation for t_x and we immediately deduce (2).

To prove (3), let $\xi(x, \lambda)$ be the solution of (2.5.11) with $\xi(0) = 0$. Note that $\xi(1, \lambda)$ is strictly decreasing in λ, $\xi(1, \lambda) \to -\pi/2$ as $\lambda \to \infty$ and $\xi(1, \lambda) \to \infty$ as $\lambda \to -\infty$. Also, note that $\xi(1, 0) = \theta(u_0)$ and that λ is an eigenvalue if and only if $\xi(1, \lambda) = k\pi$ for some integer $k \geq 0$. Since $\dim W^u(u_0)$ is the number of positive eigenvalues of the linear variational operator, this gives the proof of (3). The proof of the theorem is complete.

Remark 2.5.3. An equilibrium point of (2.5.5), (2.5.6) is hyperbolic and stable if and only if $\theta(u_0) < 0$.

It also is possible to obtain a characterization of the equilibrium points by the consideration of an initial value problem at $x = 1$ and integrating backwards. In fact, consider the initial value problem

$$\begin{aligned} b^2 u_x &= v, \quad v_x = -f(x, u), \quad 0 < x < 1, \\ u(1) &= u_0, \quad v(1) = 0, \end{aligned} \tag{2.5.12}$$

where $u_0 \in \mathbb{R}$. Let $((\bar{u}(x, u_0), \bar{v}(x, u_0))$ be the solution of (2.5.12) and suppose it is defined on $[0, 1]$ for each u_0. The set E of equilibrium solutions of (2.5.5), (2.5.6) coincides with those solutions of (2.5.8) for which $\bar{v}(0, u_0) = 0$. If we let $\bar{\S} \subset \mathbb{R}^3$ be defined by $\bar{\S} = \{(\bar{u}(x, u_0), \bar{v}(x, u_0), x) : 0 \leq x \leq 1, u_0 \in \mathbb{R}\}$ and let $\bar{\S}_x = \{(\bar{u}, \bar{v}) : (\bar{u}, \bar{v}, x) \in \S\}$ be the cross-section of $\bar{\S}$ at x, then $\bar{u}(\cdot, u_0) \in E$ if and only if

$$(\bar{u}(1, u_0), \bar{v}(1, u_0)) \subset \bar{\S}_1 \cap \{(\bar{u}, 0, 1) \in \mathbb{R}^3\}.$$

More generally, u_0 corresponds to an equilibrium point if and only if $u_0 \in \S_x \cap \bar{\S}_x$ for $x \in [0, 1]$. One can introduce the angle $\bar{t}$ for the linear variational equation for (2.5.12) and prove the following

THEOREM 2.5.7.

(1) *The set of equilibria of (2.5.5), (2.5.6) is in one-to-one correspondence with the set* $\S_x \cap \overline{\S}_x$ *for* $x \in [0,1]$.
(2) *An equilibrium point corresponding to* u_0 *is hyperbolic if and only if* $\S_x$ *is transversal to* $\overline{\S}_x$ *at* u_0 *for every* $x \in [0,1]$.
(3) *If* u_0 *corresponds to an hyperbolic equilibrium point and* $W^u(u_0)$ *is the unstable manifold, then*

$$\dim W^u(u_0) = 1 + [\phi(u_0, x)/\pi], \quad x \in [0,1],$$

where $\phi(u_o, x) = t(x, u_0) - \bar{t}(x, u_0)$ *is the angle between* $\S_x$ *and* $\overline{\S}_x$ *measured clockwise.*

2.6. Spatially dependent nonlinearity

In this section, we consider a reaction diffusion equation with constant but small diffusion and a nonlinearity for which the zeros are spatially dependent. The objective is to determine the transition layers of the equilibrium solutions and to classify the stability properties of these solutions in terms of the zeros of the nonlinearity. In order to keep the notation and details to a minimum, we will restrict the nonlinearity to be a cubic with only the middle zero being spatially dependent. The existence of stable solutions in such situations was observed by PELETIER (1976), (1978). With special restrictions on the middle zero, CLÉMENT and PELETIER (1985) established the existence and stability of a strictly monotone equilibrium solution. The characterization of all stable solutions under generic conditions on the middle zero was given by Angenent, MALLET-PARET and PELETIER (1987) when the middle zero is a smooth function and by ROCHA (1988) when it is a step function. The methods used by ROCHA (1988) for step functions lead to characterization of the stability properties of all solutions, whereas the methods of Angenent, MALLET-PARET and PELETIER (1987) in the smooth case can be applied only to stable solutions. The other transition layer solutions in the smooth case were considered by HALE and SAKAMOTO (1988), a paper in which they also showed how the above restrictions on the nonlinearity could by eliminated. KURLAND (1983) has described a number of the possible forms of solutons with transition layers in such situations and used the Conley index to show these forms indeed occur, but stability was not considered.

2.6.1. Smooth dependence on the spatial variable We consider the equation

$$u_t = \varepsilon^2 u_{xx} + f(x, u) \quad \text{in } (0, 1), \tag{2.6.1}$$

with the boundary conditions (2.3.2), $\varepsilon > 0$ a small parameter and the nonlinear function f given explicitly by

$$f(x, u) = u(1-u)[u - c(x)], \tag{2.6.2}$$

where c is a C^1-function satisfying the following conditions:

$$
\begin{aligned}
& 0 < c(x) < 1, \\
(2.6.3) \qquad & c'(\zeta) \neq 0 \text{ if } c(\zeta) = 1/2, \\
& c(x) \neq 1/2,\ c'(x) \neq 0 \text{ for } x = 0 \text{ and } 1.
\end{aligned}
$$

The following result is due to Angenent, Mallet-Paret and Peletier (1987).

THEOREM 2.6.1. *Let* $\Sigma_0 = \{\,\zeta : c(\zeta) = 1/2\,\}$ *and let* $\Sigma \subset \Sigma_0$ *be the sequence* $0 < \zeta_1 < \zeta_2 < \ldots < \zeta_M < 1$. *Then there is an* $\varepsilon_0 > 0$ *such that, for* $0 < \varepsilon < \varepsilon_0$, *there is a stable equilibrium solution* $\varphi^\varepsilon(x)$ *of (2.6.1), (2.3.2) which is monotone in a neighborhood of each* ζ_j, $c'(\zeta_j)\varphi_x^\varepsilon(\zeta_j) < 0$ *for each* j *and* $\varphi^\varepsilon(x) \to 0$ *or* 1 *as* $\varepsilon \to 0$ *uniformly on any closed interval not containing* Σ. *The number of such solutions is the* M^{th}*-Fibonacci number and all stable solutions are obtained in this way.*

The Fibonacci numbers are defined recursively by the relations $k_0 = 2, k_1 = 3$, $k_N = k_{N-1} + k_{N-2}$.

REMARK 2.6.1. In Theorem 2.3.1, we have seen that, for every $\varepsilon > 0$, the only stable equilibria of (2.6.1), (2.3.2) with $c(x) \equiv \frac{1}{2}$ are constant functions. Now suppose that $c(x) = \frac{1}{2} + \nu(x - \frac{1}{2})$, where $0 < \nu < 1$. Theorem 2.6.1 asserts that, regardless of the size of $\nu \in (0,\,1)$, there is an ε so that (2.6.1), (2.3.2) has a stable nonconstant equilibrium; that is, even small smooth perturbations in f do not preserve the number of stable equilibria uniformly with respect to ε.

We give an outline of the proof for the case in which $\S_0 = \{\zeta\}$; that is, a single point ζ. For definiteness, we also suppose that $c'(\zeta) < 0$. Let $\Omega_0 = (0, \zeta)$, $\Omega_1 = (\zeta, 1)$, $\Omega_0^\delta = (0, \zeta - \delta)$, $\Omega_1^\delta = (\zeta + \delta, 1)$ for $\delta > 0$. The first step is to construct an upper solution u^+ and lower solution u_- such that $u_- \leq u^+$ in $\bar{\Omega}$ which are close to zero on Ω_0^δ and close to 1 on Ω_1^δ with a sharp transition layer in the interval $(\zeta - \delta, \zeta + \delta)$. The existence of a stable solution in $[u_-, u^+]$ is then a consequence of Theorem 2.5.3. This stable solution clearly is nonconstant. The most difficult part of the proof is to show that there is only one stable solution in the interval $[u_-, u^+]$. This is accomplished by a careful analysis of the eigenvalue problem for the linearization about the solution and showing that there is an $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, the principal eigenvalue is negative for any equilibrium solution in the interval $[u_-, u^+]$. If we let μ_ε be this principal eigenvalue, then a nontrivial computation shows that

$$
\begin{aligned}
\mu_\varepsilon &= -|K(\zeta)|\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0, \\
(2.6.4) \qquad K(\zeta) &= \frac{\int_0^1 f_x(\zeta, s)\,ds}{\int_{-\infty}^{\infty} U'(y)^2\,dy} = \frac{1}{6}\,\frac{c'(\zeta)}{\int_{-\infty}^{\infty} U'(y)^2\,dy},
\end{aligned}
$$

where U is the solution of the equation

$$
\begin{aligned}
&U_{yy} + u(1-u)(u-\tfrac{1}{2}) = 0, \\
&\lim_{y\to -\infty} U(y) = 0, \quad \lim_{y\to\infty} U(y) = 1.
\end{aligned} \tag{2.6.5}
$$

The last step of the proof is to show that every stable solution can be obtained in this way.

Theorem 2.6.1 is concerned only with stable solutions and the method of proof does not permit the discussion of other types of solutions. Hale and Sakamoto (1988) have shown that it is possible to give a generalization of part of Theorem 2.6.1, at least that part that is concerned with the existence and index of solutions that have one transition layer at each of the points at which the function c is equal to $1/2$. To state the result, we need some notation. As in Theorem 2.6.1, we let $\S \subset \S_0$ be the sequence $0 < \zeta_1 < \cdots < \zeta_M < 1$ and define

$$
\begin{aligned}
\Omega_0 \quad &= \begin{cases} [0,\zeta_1) \cup (\zeta_2,\zeta_3) \cup \cdots \cup (\zeta_M, 1] & \text{if M is even} \\ \\ [0,\zeta_1) \cup (\zeta_2,\zeta_3) \cup \cdots \cup (\zeta_{M-1}, \zeta_M) & \text{if M is odd} \end{cases} \\
\\
\Omega_1 \quad &= [0,1] \setminus \bar{\Omega}_0
\end{aligned} \tag{2.6.8}
$$

Theorem 2.6.2. *Let $\S, \Omega_0, \Omega_1$ be defined as above and let $K(\zeta)$, U, be defined by (2.6.4), (2.6.5). There exist an $\varepsilon_0 > 0$, μ_0, $d_1, 0 < d_1 < d_0 = \frac{1}{2}\min\{\zeta_1, \zeta_{j+1} - \zeta_j, 1 \le j \le M-1, 1-\zeta_M\}$, and two families of equilibrium solutions $\varphi^\varepsilon_\pm(x)$ of (2.6.1), (2.6.2) for which the following properties hold for $0 < \varepsilon \le \varepsilon_0$:*

(1) *Uniformly on compact subsets,*

$$
\lim_{\varepsilon\to 0} \varphi^\varepsilon_+(x) = \begin{cases} 0 & \text{on } \Omega_0 \\ 1 & \text{on } \Omega_1 \end{cases}
$$

$$
\lim_{\varepsilon\to 0} \varphi^\varepsilon_-(x) = \begin{cases} 1 & \text{on } \Omega_0 \\ 0 & \text{on } \Omega_1 \end{cases}
$$

(2) *The first M eigenvalues $\lambda_j(\varepsilon, \pm)$, $j = 1, 2, \ldots, M$, of the linear variational operator about $\varphi^\varepsilon_\pm(x)$ are given by*

$$
\begin{aligned}
\lambda_j(\varepsilon, +) \quad &= (-1)^j K(\zeta_j)\varepsilon + o(\varepsilon) \\
\lambda_j(\varepsilon, -) \quad &= (-1)^{j+1} K(\zeta_j)\varepsilon + o(\varepsilon)
\end{aligned}
$$

as $\varepsilon \to 0$ and the remaining eigenvalues are $\le -\mu_0$.

(3) *$\varphi^\varepsilon_\pm(x)$ are monotone over the intervals $[\zeta_j - \varepsilon d_1, \zeta_j + \varepsilon d_1]$, $j = 1, 2, \ldots, M$. More precisely,*

$$
\pm(-1)^{j+1}\varphi^\varepsilon_{\pm,x}(x) > 0 \text{ for } x \in [\zeta_j - \varepsilon d_1, \zeta_j + \varepsilon d_1], j = 1, 2, \ldots, M.
$$

Hale and Sakamoto (1988) have given a generalization of Theorem 2.6.2 to the situation where the function f need not be a cubic. The principal hypothesis on f is that the phase portrait of the solutions of the equation $v_{xx} + f(\zeta_j, v) = 0$ has a heteroclinic orbit from 0 to 1 for each j and that this orbit will not exist if we replace ζ_j by a constant close to but not equal to ζ_j.

It is possible to prove this result in several different ways. For the existence of such solutions, we can follow the procedure in Fife (1974), (1976), Ito (1984), Mimura, Tabata and Hosono (1980). This involves taking first an approximation of the solution to be one that consists of a continuous function which is approximately equal to the solution $U((x - \zeta_j)/\varepsilon)$ of (2.6.5) (or its reflection) in a neighborhood of each of the points ζ_j, $j = 1, 2, \dots, M$, and is 0 or 1 on the other parts of the interval. The Implicit Function Theorem can then be used to show the existence of an exact solution near this approximate one. After showing that the limiting value of the solution as $\varepsilon \to 0$ is a step function with values 0 or 1, the approximate eigenvalues can be shown to satisfy the statement in the theorem by employing the so called SLEP (singular limit eigenvalue problem) method (see, for example, Fujii and Nishiura (1987), Nishiura (1982)).

The method of proof given by Hale and Sakamoto (1988) has a more geometric flavor and leads to the existence and stability of the solution at the same time. More precisely, they give the detailed proof by using the method of Lyapunov-Schmidt. It is clear from the approximations and estimates given in that paper that the center manifold theorem also could be used. This latter method also gives information about the unstable manifold of the equilibrium point. To apply these more geometric methods, one must obtain a better approximation for the solution than the obvious one mentioned above. Hale and Sakamoto (1988) show how to obtain a more accurate asymptotic formula for the solution and, in fact, it approximates a solution of the equation up through terms of order ε and the differential equation is satisfied up through terms of order ε^2. This is sufficient to apply Lyapunov-Schmidt or the center manifold theorem since it is shown that the eigenvalues satisfy the properties stated in the theorem; that is, the smallest ones are of order ε. The details of any proof of this type is very technical and we refer the reader to the cited papers as well as one by Taniguchi (1996).

It seems plausible, but the detailed proof has not been given, that the only solutions with one transition layer at the points ζ_j are the ones given above.

It is known (see Kurland (1983)) that there is no upper bound on the number of equilibrium solutions as $\varepsilon \to 0$ and, in fact, as $\varepsilon \to 0$, there are more and more highly oscillatory solutons. The index of these solutions is not known. Such information plays a very important role in the understanding of complete flow on the attractor since this is determined by the unstable manifolds of the equilibrium points.

2.6.2. Piecewise constant spatial dependence. ROCHA (1988) has considered (2.6.1), (2.3.2) in the situation where the function c is a step function. He has obtained a complete analogue of Theorem 2.6.1. Also, the method of proof permits the discussion of unstable equilibrium points and their index as well as information about the connections between the equilibria. The complete flow on the attractor has been analyzed in some nontrivial cases for all $\varepsilon > 0$. Due to the fact that the middle zero of f is a step function and not constant, the number of equilibria is bounded independent of ε!

We now state the result of Rocha (1988).

THEOREM 2.6.3. *Let $c : [0,1] \to (0,1)$ be a step function with $c(x) \neq 1/2$ for any x. Let c jump across $1/2$ at the M points $0 < \zeta_1 < \zeta_2 < \cdots < \zeta_M < 1$ and let $\zeta_0 = 0, \zeta_{M+1} = 1$. Then there is an $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, the number of stable equilibrium points of (2.6.1), (2.6.2), (2.3.2) is exactly the M^{th} Fibonacci number. Furthermore, a stable solution $\varphi^\varepsilon(x) \to 0$ or 1 on each interval (ζ_j, ζ_{j+1}), $j = 0, 1, \ldots, M$, and is monotone in a neighborhood of each ζ_j with*

$$[c(\zeta_j^+) - c(\zeta_j^-)]\varphi_x^\varepsilon(\zeta_j) < 0, \quad j = 1, 2, \ldots, M.$$

The proof of this result uses the method discussed in Section 2.5.3 with $b^2 = \varepsilon^2$ and the notation $u(x, u_0)$, $\theta(u_0)$, $t(x, u_0)$, $\bar{t}(x, u_0)$ of that section. It is first shown that, for any $0 < \delta < 1/2$, there is an $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, each equilibrium point $\varphi = \varphi(\cdot, u_0)$ with $u_0 \in [\delta, 1-\delta]$ is unstable. This permits one to reduce the discussion to the manner in which the stable and unstable manifolds of 0 and 1 intersect for ε small, which is an argument in the phase plane.

ROCHA (1988) has given the complete attractor for special situations in which there are one or two jumps of the function c across $1/2$.

2.7. Connecting orbits.

In this section, we present general results of FIEDLER and ROCHA (1995) on the existence of connecting orbits between equilibrium points of the scalar equation

$$\begin{aligned} u_t &= a(x)u_{xx} + f(x, u, u_x), \quad x \in (0,1), \\ u_x &= 0 \quad \text{at } x = 0, 1, \end{aligned} \tag{2.7.1}$$

where $a \in C^2([0,1])$ is positive and f is a C^2-function. For the space of initial data, we choose X as those functions v in $H^2([0,1])$ with $v_x = 0$ at $x = 0, 1$. Equation (2.7.1) defines a local semiflow in X. We also suppose that the system is dissipative in X and thus we have a compact global attractor $\mathcal{A}$.

As we have noted in Section 2.1, (2.7.1) is gradient and $\omega(\varphi)$ is a single equilibrium point; that is, a solution of the equation

$$\begin{aligned} a(x)u_{xx} + f(x, u, u_x) &= 0, \quad x \in (0,1), \\ u_x &= 0 \quad \text{at } x = 0, 1, \end{aligned} \tag{2.7.2}$$

We also assume that each element of the set E of equilibria is hyperbolic so that $\mathcal{A} = \cup_{\varphi \in E} W^u(\varphi)$, where $W^u(\varphi)$ is the unstable manifold of φ and E is the set of equilibria.

Given $\varphi, \psi \in E$, let $C(\varphi, \psi)$ denote the set of *connecting orbits* from φ to ψ; that is, the set of trajectories $u(t)$ which are defined for $t \in \mathbb{R}$ with the property that $\omega(u(\cdot)) = \psi$ and $\alpha(u(\cdot)) = \varphi$. It is clear that

$$\mathcal{A} = E \cup (\cup_{\varphi, \psi \in E} C(\varphi, \psi)),$$
$$C(\varphi, \psi) = W^u(\varphi) \cap W^s(\psi).$$

Since $\mathcal{A}$ is compact, the set E is finite, say $E = \{\varphi_1, \ldots, \varphi_k\}$. Let us order these elements so that

$$\varphi_1(0) < \varphi_2(0) < \cdots < \varphi_k(0). \tag{2.7.3}$$

By uniqueness of the initial value problem for the ODE (2.7.2), these values are distinct. At the other boundary point $x = 1$, that order may have changed and we have a permutation π of the integers $1, 2, \ldots, k$ defined by

$$\varphi_{\pi(1)}(1) < \varphi_{\pi(2)}(1) < \cdots < \varphi_{\pi(k)}(1). \tag{2.7.4}$$

This permutation was introduced by Fusco and Rocha (1991) and some important consequences were deduced including Lemma 2.7.1 below. The amazing thing about this permutation is that it characterizes the existence of connecting orbits as stated in the following result of Fiedler and Rocha (1995).

Theorem 2.7.1. *Let π be the permutation defined by (2.7.3), (2.7.4). Then π determines, in an explicit constructive process, which equilibria are connected and which are not; that is, this permutation determines which of the sets $C(\varphi, \psi)$ are nonempty.*

The connection problem has a rather long history with contributions resulting from detailed investigation of special cases, especially the Chafee-Infante problem (see Fiedler and Rocha (1995) for a detailed discussion and bibliography). The proof of Theorem 2.7.1 is based on several constructive lemmas which also permit one to determine the index of each equilibrium as well as connecting orbits. For any function $u \in X$, let $z(u)$, the *zero number* of u, be the number of strict sign changes of u.

Lemma 2.7.1. *The permutation π constructively and explicitly determines $i(\varphi)$ and $z(\varphi - \psi)$ for all $\varphi, \psi \in E$. They are given explicitly as*

$$i(\varphi_m) = \S_{j=1}^{m-1} (-1)^{j+1} \operatorname{sign} (\pi^{-1}(j+1) - \pi^{-1}(j)) \tag{2.7.5}$$

for $1 \le m \le k$ and

$$\begin{aligned} z(v_n - v_m) = i(v_m) \quad &+ \tfrac{1}{2}[(-1)^n \operatorname{sign} (\pi^{-1}(n) - \pi^{-1}(m)) - 1] \\ &+ \S_{j=m+1}^{n-1} (-1)^j \operatorname{sign} (\pi^{-1}(j) - \pi^{-1}(m)) \end{aligned} \tag{2.7.6}$$

for $1 \leq m < n \leq k$. These numbers are determined by the recursive formulas:

$$i(\varphi_{m+1}) = i(\varphi_m) + (-1)^{m+1}\,\mathrm{sign}\,(\pi^{-1}(m+1) - \pi^{-1}(m)), \tag{2.7.7}$$

$$i(\varphi_1) = 0 = i(\varphi_k), \tag{2.7.8}$$

$$z(\varphi_{m+1} - \varphi_m) = \min\{i(\varphi_m), i(\varphi_{m+1})\}, \tag{2.7.9}$$

$$\begin{aligned} z(\varphi_{n+1} - \varphi_m) &= z(\varphi_n - \varphi_m) \\ &\quad + \tfrac{1}{2}[(-1)^{n+1}\,\mathrm{sign}\,(\pi^{-1}(n+1) - \pi^{-1}(m)) \\ &\quad + (-1)^n\,\mathrm{sign}\,(\pi^{-1}(n) - \pi^{-1}(m)))] \end{aligned} \tag{2.7.10}$$

$$z(\varphi_n - \varphi_1) = z(\varphi_k - \varphi_m) = 0. \tag{2.7.11}$$

The second step of the proof is to observe that the existence of $C(\varphi, \psi)$ implies a very special type of cascading.

Lemma 2.7.2. (*Cascading*) *Suppose that $\varphi, \psi \in E$ and let $n = i(\varphi) - i(\psi)$. Then $C(\varphi, \psi) \neq \emptyset$ if and only if there is a cascade $\{\varphi = e_0, e_1, \ldots, e_n = \psi\}$ of equilibria such that, for all $0 \leq j < n$, we have*

(i) $i(e_{j+1}) = i(e_j) + 1$,
(ii) $C(e_{j+1}, e_j) \neq \emptyset$.

By the cascading lemma, it is sufficient to check all of the possible connections from φ to ψ when $i(\varphi) = i(\psi) + 1$. There are some criteria for determining if such connections exist. To state this precisely, we introduce the following definition.

Definition 2.7.1. If $\varphi, \psi \in E$, $i(\varphi) = i(\psi) + 1$, we say that connections between φ and ψ are *blocked* if one of the following conditions holds:

(i) $z(\varphi - \psi) \neq i(\psi)$,
(ii) there is a $\eta \in E$ with $\eta(0)$ between $\varphi(0)$ and $\psi(0)$ such that $z(\varphi - \eta) = z(\psi - \eta) = z(\varphi - \psi)$.

It was observed by Brunovský and Fiedler (1989) that blocking prevents connections. The reverse statement is due to Fiedler and Rocha (1995) and stated as

Lemma 2.7.3. (*Liberalism*) *If $\varphi, \psi \in E$, $i(\varphi) = i(\psi) + 1$ and connections from φ to ψ are blocked, then $C(\varphi, \psi) = \emptyset$. If they are not blocked, then $C(\varphi, \psi) \neq \emptyset$.*

The proofs of these results are very long and depend in a significant manner upon the fact that the zero number of solutions of linear parabolic equations drop immediately after the existence of a double zero (Angenent (1988)), the fact that stable and unstable manifolds always intersect transversally (Henry (1985), Angenent (1986)), phase space analysis, the Sturm-Liouville theory and the Conley index.

Let us consider two differential equations (2.7.1) corresponding to (a, f) and (b, g) with attractors $\mathcal{A}_{(a,f)}$ and $\mathcal{A}_{(b,g)}$ and equilibrium sets $E_{(a,f)}$ and $E_{(b,g)}$. We say that $\mathcal{A}_{(a,f)}$ and $\mathcal{A}_{(b,g)}$ are *topologically equivalent* if there is a homeomorphism $h : \mathcal{A}_{(a,f)} \to \mathcal{A}_{(b,g)}$ which takes orbits of (a, f) to orbits of (b, g) and preserves the sense of direction in time.

We state the following result of Fiedler and Rocha (1996) without comments on the proof.

THEOREM 2.7.2. *Assuming that (a, f) and (b, g) are dissipative and all equilibrium points are hyperbolic, then $\pi_{(a,f)} = \pi_{(b,g)}$ implies $\mathcal{A}_{(a,f)}$ and $\mathcal{A}_{(b,g)}$ are topologically equivalent.*

There are many problems remaining in this area. For example, it would be very interesting to understand which permutations can be realized by an equation of the form (2.7.1). This would be especially important in modeling. These questions are discussed at some length in FIEDLER and ROCHA (1995) (see also FIEDLER (1994)).

EXAMPLE 2.7.1. Let us apply this result to the classical Chafee-Infante problem with symmetric reaction term:

$$u_t = \varepsilon^2 u_{xx} + u - u^3 \quad \text{in } (0, 1) \tag{2.7.12}$$

with the homogeneous Neumann boundary conditions. For $\varepsilon^{-1} \in (n\pi, (n+1)\pi)$, we have remarked earlier that CHAFEE and INFANTE (1974)) have shown that there are exactly $k = 2n + 1$ equilibrium points $u_0 = 0, u_j^{\pm}, j = 1, 2, \ldots, n - 1$, $u_n^{\pm} = \pm 1$, each of which is hyperbolic. Also, $i(u_0) = n$, $i(u_j^{\pm}) = n - j$, $j = 1, 2, \ldots, n$, the attractor $\mathcal{A}_n$ has dimension n, is the closure of $W^u(0)$ and $C(u_{j+1}^{+}, u_j^{+}) \neq \emptyset$ for any j (Henry (1985)). Even more is known about the dimension of the sets of connecting orbits. These results were obtained using bifurcation theory, spectral analysis and transversality theory.

Let us show how the general theory above allows us to easily conclude information about connecting orbits. To be specific, let us fix $k = 7$; that is, $n = 3$, and fix $\varepsilon^{-1} \in (3\pi, 4\pi)$. The attractor then has exactly 7 hyperbolic equilibrium points. We order them according to their values at $x = 0 : \varphi_1(0) < \cdots < \varphi_7(0)$. The functions $\varphi_2 = -\varphi_4$ have one zero, the functions $\varphi_3 = -\varphi_5$ have two zeros and the others are constants. Because of the symmetry in the problem, we also see immediately that

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 71 & 6 & 3 & 4 & 5 & 2 & 7 \end{bmatrix}.$$

If we use the recursive formulas (2.7.6), (2.7.7), then we have the vector of indices

$$(i(\varphi_n))_n = [0, 1, 2, 3, 2, 1, 0],$$

the same result that we noted above obtained in a more analytical way.

To check for connections, we must check for blockage and thus we must determine properties of the zeros of the differences between equilibria. Using the recursive formulas (2.7.9)–(2.7.11), we deduce that the matrix of these differences is given by

$$(z(\varphi_n - \varphi_m))_{n,m} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where we have put the index $i(v_n)$ on the diagonal for convenience. The off diagonal elements in the matrix are given explicitly by $z(\varphi_n - \varphi_m) = 3 - max\{|4-m|, |4-n|\}$.

From the definition of blockage, Lemma 2.7.3, the matrix for the indices and the matrix for the $z(\varphi_n - \varphi_m)$, it is easy to check that there is no blockage if $i(\varphi) = i(\psi) + 1$ and thus there is connection between these points. This gives us the following graphical information about the one-dimensional connections: 4 goes to 3 and 5, which go to 2 and 6 which go to 1 and 7.

If we take into account the indices of the respective points and use continuous dependence of solutions on the data, then we can also see that 4 connects to each point and 3 and 5 connect to $2, 6, 1, 7$.

In this way, we obtain the qualitative information of the flow that was mentioned above by very simple computations.

For other interesting examples with 11 equilibrium points, see Fiedler and Rocha (1995) and for a classification of the possible one dimensional connection graphs with 9 or fewer equilibrium points, see Fiedler (1994).

2.8. Nonlocal reaction terms

We have seen in previous sections how stable nonconstant equilibrium solutions can occur if either the diffusion coefficient or the nonlinear reaction term depends upon the spatial variable. In either situation, it was not possible to obtain a bifurcation from a stable constant solution to a stable nonconstant solution. In this section, we discuss the generation of such solutions as a consequence of the presence of nonlocal terms in the equation. In the classical Sturm-Liouville theory of self adjoint eigenvalue problems, the eigenvalues are ordered according to the number of zeros of the eigenfunctions. As we will see, if there are nonlocal terms in a linear equation, then this type of ordering may not hold. Therefore, it is possible to have a bifurcation from a stable constant equilibrium solution to a stable nonconstant equilibrium solution. There are many applications for equations of this type; for example, ballast resistors, population dynamics, etc. We will see also that they occur as limiting situations for pairs of reaction diffusion equations where one of the diffusion coefficients is very large.

As the simplest illustration of equations with nonlocal terms, we consider

$$u_t = u_{xx} + f(u, [u]) \quad \text{in } (0, 1), \tag{2.8.1}$$

with the homogeneous Neumann boundary condition (2.3.2), where $f(u, v)$ is a C^2-function and there are a continuous function γ on $(0, 1)$ and a C^2-function r on R such that

$$[\varphi] = \int_0^1 \gamma(y) r(\varphi(y))\, dy \tag{2.8.2}$$

for all $\varphi \in H^1((0, 1))$. If $\gamma(y) \equiv 1$ for all y and $r(u) = u$ for all $u \in R$, then $[\varphi]$ is just the average of φ.

Chafee (1981) encountered an equation of this type in his study of the ballast resistor. This is a device consisting of a straight segment of very thin wire surrounded by a gas having a fixed temperature. For the ballast resistor with constant voltage differential V along a wire of length one, a reasonable model asserts that the temperature in the wire satisfies the equation

$$u_t = u_{xx} - g(u) + V^2 r(u)[\int_0^1 r(u(y, t))\, ds]^{-2} \quad \text{in } (0,1) \tag{2.8.3}$$

with homogeneous Neumann boundary conditions, where $r(u)$ is the resistance in the wire. Let us consider only nonnegative solutions of (2.8.3). Under reasonable physical conditions, Chafee (1981) showed that there is a unique constant solution c. Furthermore, there is a V^* such that, for $V < V^*$, every solution approaches c as $t \to \infty$ and c is stable. From the theory of Chapter 1, this implies that the global attractor is the point c. For $V > V^*$, the point c is unstable and there is a stable nonconstant solution which arises as a primary bifurcation from the point c. The complete dynamics of (2.8.3) is not known.

In the modeling of population dynamics, it is sometimes reasonable to suppose that some weighted average of the total population affects an individual. In such a situation, the model could lead to equation (2.8.1) (see, for example, Levin and Segal (1982), (1985)) and stable nonconstant equilibrium solutions can arise through primary bifurcations from constant equilibrium states. A typical example is

$$u_t = du_{xx} + u[1 + a\int_0^1 \alpha(x-y)u(y, \cdot)\, dy - b\int_0^1 \beta(x-y)u(y, \cdot)\, dy], \tag{2.8.4}$$

where all constants and functions are positive and Neumann boundary conditions are imposed. There are many other interesting examples in the papers of

Levin and Segal, including systems of equations. Calsina and Perelló (1995) have scalar models with nonlocal terms for populations competing amongst themselves but with different growth rates and systems of equations when a new species is introduced and competing for the same resource.

To have a primary bifurcation from a constant solution lead to a stable nonconstant equilibrium solution is in complete contrast to the situation that occurs without the term $[u]$. In fact, suppose that f in (2.8.1) is a function only of u and consider the linear variational equation about a constant equilibrium solution u_0. If the eigenvalues of the corresponding linear operator are ordered as $\mu_1 > \mu_2 > \ldots$, then the corresponding eigenfunctions $\varphi_1, \varphi_2, \ldots$ have the property that φ_j has exactly $j-1$ zeros in the interval (0, 1). If $f(u) = f_\lambda(u)$ depends upon a parameter λ, then the eigenvalues $\mu_j = \mu_j(\lambda)$ depend upon λ. If we suppose that u_0 is stable (and hyperbolic) for $\lambda < \lambda_0$ and unstable (and hyperbolic) for $\lambda > \lambda_0$, then $f'_\lambda(u_0) = 0$ and $\mu_1(\lambda_0) = 0$. The corresponding eigenfunction satisfies $u_{xx} = 0$ or $u_x =$ constant. The Neumann boundary conditions imply that $u_x = 0$ and so $u =$ constant. Thus, any bifurcation will be in the direction of constant functions and one will not obtain a stable nonconstant equilibrium solution. As a consequence, the stable nonconstant equilibrium solutions, if they exist, cannot occur as primary bifurcations from constant solutions.

We now give a simple explanation of why the existence of nonlocal terms in (2.8.1) can lead to stable nonconstant equilibrium solutions through primary bifurcations from constant solutions. We first assume that $\gamma(y) \equiv 1$ for all y and $r(u) = u$ for all $u \in R$; that is, $[\varphi] = \bar{\varphi} = \int_0^1 \varphi(y)\,dy$. The eigenvalues μ of the linear variational operator for a constant solution u_0 of (2.8.1) are the solutions of the boundary value problem

$$\text{(2.8.5)} \qquad \begin{aligned} v_{xx} + \alpha v &+ \beta \int_0^1 v(y)\,dy = \mu v, \quad 0 < x < 1, \\ v_x &= 0 \text{ at } x = 0,\ 1, \end{aligned}$$

where $\alpha = f_u(u_0,\, u_0)$, $\beta = f_{[u]}(u_0,\, u_0)$. By expanding $v(x)$ as a Fourier series,

$$v(x) = u_0 + \frac{1}{2}\Sigma_{n\geq 1} v_n cos\, n\pi x,$$

we observe that the eigenvalues μ_n and the eigenfunctions φ_n are given by

$$\mu_0 = \alpha + \beta, \quad \varphi_0(x) = 1,$$

$$\mu_n = -n^2\pi^2 + \alpha, \quad \varphi_n(x) = \frac{1}{2} cos\, n\pi x,\ n \geq 1.$$

It always is true that $\mu_1 > \mu_2 > \ldots$; that is, these eigenvalues are ordered according to the number of zeros in (0, 1) of the corresponding eigenfunctions. On the other hand, μ_0 cannot be ordered with the other eigenvalues. Its relationship to the other eigenvalues depends upon the size of β. The sign of the largest eigenvalue depends upon the relative size of α and β. If we are interested in the first bifurcation that will occur from a stable equilibrium u_0, then the bifurcation will be to a constant solution if $\mu_0 > \mu_1$ and it will be a nonconstant solution if $\mu_0 < \mu_1$. In the $(\alpha,\, \beta)$-plane, the line $\beta = -\pi^2$ corresponds to the situation where $\mu_0 = \mu_1$, the half ray $\alpha = -\beta,\ \alpha < \pi^2$ corresponds to the curve of bifurcation to a constant solution, and the half ray $\alpha = \pi^2,\ \beta < -\pi^2$ corresponds to the curve of bifurcation to nonconstant solutions. If these bifurcations are supercritical, then we obtain stable solutions after bifurcation. On the half ray $\alpha = \pi^2, \beta < -\pi^2$, the primary bifurcation leads to stable nonconstant solutions.

Let us now consider the general case (2.8.2). If we define $\tilde{\gamma}(s) = \gamma(s)r'(u_0)$, then the linear variational equation about a constant solution u_0 is

$$\begin{aligned} v_{xx} + \alpha v \quad &+\beta \int_0^1 \tilde{\gamma}(y)v(y)\,dy = 0, \quad 0 < x < 1, \\ v_x \quad &= 0 \text{ at } x = 0,\, 1, \end{aligned} \tag{2.8.6}$$

The eigenvalues and eigenfunctions of the linear operator defined by this equation are

$$\mu_0 = \alpha + \beta \int_0^1 \tilde{\gamma}(s)\,ds, \quad \psi_0(x) = 1,$$

$$\mu_n = \alpha - n^2\pi^2, \quad \psi_n(x) = \cos n\pi x - \frac{\beta}{n^2}\int_0^1 \tilde{\gamma}(s)\cos n\pi s\,ds,\ n \geq 1.$$

The situation is essentially the same as before.

If we return to the equation for the ballast resistor (2.8.3), then the linear variational equation about the constant solution is a special case of (2.8.6) and we have an explanation of the appearance of a stable nonconstant equilibrium solution.

By adding functional dependence (nonlocal terms) in the equation as above, we are allowing the vector field to give special emphasis to the (weighted) projection of the solution onto the linear subspace spanned by the first eigenfunction of the operator $\partial^2/\partial x^2$ with Neumann boundary conditions; that is, the constant functions. By projecting onto larger subspaces, one can obtain very complicated dynamics from a scalar equation. As a simple illustration, consider the functional

$$f(x,\, \varphi) = g(\varphi_1,\, \varphi_2) + h(\varphi_1,\, \varphi_2)\cos \pi x$$

$$\varphi_1 = \int_0^1 \varphi(y)\,dy, \quad \varphi_2 = 2\int_0^1 \varphi(y)\cos \pi y\,dy$$

and the equation

$$u_t = u_{xx} + f(x,\, u(\cdot,\, t)), \qquad 0 < x < 1,$$

with homogeneous Neumann boundary conditions. If $(a(t),\, b(t))$ is a solution of the equation

$$\dot{a} = g(a,\, b), \qquad \dot{b} = -\pi^2 b + h(a,\, b),$$

then $u(x, t) = a(t) + b(t) cos\, \pi x$ is a solution of the PDE. Thus, for any given dynamics in the plane, there is a scalar one dimensional parabolic equation for which the dynamics contains the one specified in the plane. We emphasize that to obtain this result, we allowed the vector field to depend explicitly upon the spatial variable.

By projecting onto the span of more of the eigenfunctions, it is possible to reproduce any dynamics in R^n. Of course, the nonlinearities obtained in this way are artificial and may not correspond to any meaningful physical problem. On the other hand, such a discussion makes it clear that any problem which involves nonlocal terms has the possibility of exhibiting complicated dynamics.

Fiedler and Polàčik (1990) recently have observed how complicated dynamics can be achieved with f depending on the spatial variables and linear terms involving the nonlocal terms. More specifically, they consider the equation

$$u_t = u_{xx} + a(x)u + g(x, u) + c(x)\alpha(u), \quad 0 < x < 1, \tag{2.8.7}$$

where

$$\alpha(u) = \int_0^1 \nu(y)u(y, \cdot)\, dy \tag{2.8.8}$$

a, c are continuous, $\nu \in L^2(0, 1)$, g is C^∞ in u and the solution u of the equation is required to satisfy homogeneous Dirichlet boundary conditions (Neumann conditions probably could be used). Their main result is the following

Theorem 2.8.1. *Consider (2.8.7), (2.8.8) with homogeneous Dirichlet boundary conditions. There is a residual set $\mathcal{G} \subset C[0, 1]$ such that, for any $a \in \mathcal{G}$, the following holds. Let $\omega_1, \omega_2, \ldots, \omega_m$ be given mutually distinct positive numbers. Let V be any polynomial vector field on R^{2m} of degree N such that $V(0) = 0$ and $V'(0) = 0$. Then there exist c, $\nu \in C[0, 1]$ and a function g which is C^∞ in u which vanishes together with its first derivative at $u = 0$, such that*

(i) The spectrum of the linearized operator $Au = -u_{xx} - a(x)u - c(x)\alpha(u)$ is given by the algebraically simple eigenvalues

$$+i\omega_j,\ -i\omega_j,\ j \le m,\ \lambda_k,\ k > 2m,$$

where $\lambda_k \neq 0$ is the k^{th} eigenvalue of the unperturbed operator $Lu = -u_{xx} - a(x)u$.

(ii) If H is the vector field on a C^N-center manifold of the equilibrium $u = 0$ of equation (2.8.7), (2.8.8), then, in real coordinates, the Taylor expansion of H at 0 coincides with V for orders 2 to N.

We remark that the existence of complicated dynamics obtained in the above way involves the consideration of vector fields with non local terms and also explicit dependence on the spatial variable.There still remains the problem of deciding if a particular equation can exhibit complicated dynamics and, in particular, how complicated the dynamics can be if we do not allow explicit spatial dependence.

For a system of two reaction diffusion equations for which one diffusion coefficient is large relative to the other, it is sometimes possible for the flow on the attractor to be reduced to a single equation with nonlocal dependence. In general, such an equation also will involve hereditary effects, but there are situations where it is reasonable for these to be neglected. Let us discuss this in more detail. Consider the system of equations

$$\begin{aligned} u_t &= du_{xx} + f(u, v), \\ v_t &= \delta^{-1} v_{xx} + g(u, v), \end{aligned} \tag{2.8.9}$$

on (0, 1) with homogeneous Neumann boundary conditions. Suppose that this equation has a global attractor. If d is fixed, HALE and SAKAMOTO (1989) (see also NISHIURA (1982), FUJII and NISHIURA (1987)) have shown that there is a $\delta_0(d) > 0$ such that, for $0 < \delta \le \delta_0(d)$, the flow on the attractor is determined from the flow on the attractor for the *shadow system*

$$\begin{aligned} u_t &= du_{xx} + f(u, \xi), \\ \dot{\xi} &= \textstyle\int_0^1 g(u(y, \cdot), \xi)\, dy. \end{aligned} \tag{2.8.10}$$

Equation (2.8.10) has a global attractor $\mathcal{A}_d$. If we suppose that λ is a positive constant and

$$g(u, v) = -\lambda[v - h(u)] \tag{2.8.11}$$

(a situation that occurs frequently in the applications), then any solution $(u(t), \xi(t))$ of (2.8.10) on the attractor $\mathcal{A}_d$ must have $\xi(t)$ bounded on $\mathbb{R}$. From the second equation in (2.8.10), $\xi(t)$ must be given by

$$\xi(t) = \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} H[u](s)\, ds, \quad H[u](t) = \int_0^1 h(u(y, t))\, dy. \tag{2.8.12}$$

This implies that u is a solution of the retarded equation with nonlocal effects

$$u_t = du_{xx} + f(u(\cdot, t), \lambda \int_{-\infty}^{0} e^{\lambda s} H[u](t+s)\, ds) \tag{2.8.13}$$

with the Neumann boundary conditions.

We remark that, if λ is large, then it is reasonable to suppose that the flow on the attractor $\mathcal{A}_d$ should be closely related to the flow defined by the equation

$$u_t = du_{xx} + f(u(\cdot, t), \int_0^1 h(u(y, t))\, dy),$$

which is a special case of (2.8.1).

FREITAS (1993) has characterized the stable solutions of (2.8.1) according to the number of zeros of the solution.

It is generally accepted folklore that traveling wave solutions of parabolic partial differential equations are representative of typical behavior of solutions of the same partial differential equation on a large unbounded domain. In this section, we present some results of Fusco, Hale and Xun (1995) which make this rigorous for a scalar reaction diffusion equation in one space variable.

If $a \in (-1, 1)$ is a given constant and

$$f_a(u) \equiv (u+a)(u^2-1) \equiv f_0(u) + ag(u), \tag{2.9.1}$$

then it is well known that the equation

$$u_t = u_{xx} - f_a(u) \qquad x \in (-\infty, \infty) \tag{2.9.2}$$

has a unique, except for translation, monotone increasing (decreasing) traveling wave solution U with wave speed $\sqrt{2}a$ and $\lim_{x\to\pm\infty} U(x) = \pm 1$ if $a > 0$ (∓ 1 if $a < 0$). For $a = 0$, both of these solutions exist and are called standing waves.

Let X be the Banach space consisting of either the space $L^p(\mathbb{R})$, $1 \le p < \infty$ or the space $C_{\text{unif}}(\mathbb{R})$ of the space of uniformly continuous functions on $\mathbb{R}$ or the space $C_0(\mathbb{R})$ of continuous functions on $\mathbb{R}$ which vanish at ∞. Let M be the one dimensional manifold in X defined by $M = \{U(\cdot + h) : h \in \mathbb{R}\}$. The traveling wave U is said to be *stable* if there is a neighborhood W of M in X such that, for any $\varphi \in W$, there is an $h_\varphi \in \mathbb{R}$ such that the solution $u(x,t)$ of (2.9.2) with $u(x,0) = \varphi(x)$ satisfies

$$\lim_{t\to\infty} |u(\cdot,t) - U(\cdot + \sqrt{2}at - h_\varphi)|_X = 0. \tag{2.9.3}$$

It is known that the traveling wave U is stable in X and also that the limit in (2.9.3) is exponential (see, for example, Henry (1981)).

Our objective is to obtain this traveling wave as the limit of solutions of the reaction diffusion equation (2.9.2) on a finite interval $(-1/\varepsilon, 1/\varepsilon)$ with homogeneous boundary conditions; that is, as the limit as $\varepsilon \to 0$ of a solution of the equation

$$u_t = u_{xx} - f_a(u) \qquad x \in (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}) \tag{2.9.4}$$

with the boundary conditions

$$u_x = 0, \qquad x = \pm\frac{1}{\varepsilon}. \tag{2.9.5}$$

Fusco, Hale and Xun (1995) have demonstrated the following result.

Theorem 2.9.1. *There exist positive constants C, a_0 such that, for any integer k, there are positive constants $\varepsilon_0, C_1, C_2, C_3$ such that, if $0 < \varepsilon \le \varepsilon_0$, $|a| \le a_0$, then, for any solution $u^\varepsilon(x,t)$ of (2.9.4), (2.9.5) with initial data in a $C\sqrt{a_0}$ L^∞-neighborhood of $U(\cdot)$, there is a positive constant h_0 such that*

$$|u(x,t) - U(x + \sqrt{2}at - h_0)| \le C_1\varepsilon^{k-2}$$

for $x \in [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$, $t \in (C_2 k \log \frac{1}{\varepsilon}, \frac{C_3}{\varepsilon})$.

The proof depends upon some very complicated estimates which force the assumption that the constant a should be bounded by a_0 which is small. Of course, it is to be expected that this restriction on a is unnecessary, but a different approach will be needed. Perhaps the more general result can be proved by exploiting the known fact that the traveling wave has the strong stability property expressed in (2.9.3) together with the fact that this limit approaches zero exponentially as $t \to \infty$.

The motivation for the proof stems from very simple considerations of the attractor $\mathcal{A}_{\varepsilon,a}$ for (2.9.4), (2.9.5). When the equilibria are hyperbolic, the attractor $\mathcal{A}_{\varepsilon,a}$ is the union of the unstable manifolds of the equilibria. There are only two stable equilibrium points ± 1 and the point a is always unstable. As $\varepsilon \to 0$, the number of equilibrium points becomes unbounded and, for any integer N, there is an interval I_N such that, if $\varepsilon \in I_N$, then there are exactly $2N+1$ equilibrium points, all hyperbolic, with two of index j, $1 \le j \le N-1$, and the equilibrium point a has index N. If φ_1 is an equilibrium point of index 1, then it is monotone on $(-1/\varepsilon, 1/\varepsilon)$ and its unstable manifold $W^u_{\varepsilon,a}(\varphi_1)$ consists of functions which also are monotone on $(-1/\varepsilon, 1/\varepsilon)$. Furthermore, φ_1 is connected to ± 1 by an orbit and the set $W^u_{\varepsilon,a}(\varphi_1)$ is a local attractor in $\mathcal{A}_{\varepsilon,a}$. Therefore, it natural to suspect that, as $\varepsilon \to 0$, this set will be related to the traveling wave.

If $a = 0$, it is not too difficult to verify the above intuitive remarks and to show that $W^u_{\varepsilon,0}(\varphi_1)$ converges to a monotone standing wave. If $a \neq 0$, the result is not obvious. To be specific, suppose that $a > 0$. Using the phase plane for the determination of the equilibria, it is not difficult to see that there is a monotone increasing equilibrium point $\varphi_{\varepsilon,a}$ of index 1 such that $\varphi_{\varepsilon,a}(y/\varepsilon) \to 1$ as $\varepsilon \to 0$ uniformly on compact subsets of $(-1,1)$. If we let $W^{u-}_{\varepsilon,a}(\varphi_{\varepsilon,a})$ designate the orbit which connects $\varphi_{\varepsilon,a}$ to -1, then it natural to suspect that each point on $W^{u-}_{\varepsilon,a}(\varphi_{\varepsilon,a})$ will approach a translate of the traveling wave of (2.9.1), (2.9.2). There also is another equilibrium solution which has index 1 and is monotone decreasing. In the scaled variables $x \mapsto y/\varepsilon$, it also approaches 1 as $\varepsilon \to 0$ uniformly on compact sets of $(-1,1)$. Therefore, each element of the orbit connecting this equilibrium point to 1 approaches the constant function 1 and does not yield a traveling wave. Similar remarks apply to the case where $a < 0$.

Theorem 2.9.1 is a precise justification of the previous intuitive remarks in the situation where a is sufficiently small. The idea for the proof is to construct an approximate manifold for the unstable manifold of the equilibrium point of index 1 and show that the traveling wave solution of (2.9.2) is nearby.

2.10. Synchronization

Suppose that we have an evolutionary partial differential equation on a bounded domain $\Omega \subset \mathbb{R}^N$ which involves diffusion with solutions defined in a Banach space $X(\Omega)$. Also, suppose that there exists a global attractor $\mathcal{A} \subset X(\Omega)$. If we think of this diffusive equation as the diffusive interaction of particles located at each position $x \in \Omega$, then it is natural to say that the system is *synchronized* if each point in the attractor is spatially independent. If this is the case, then the dynamics on the attractor is such that each partilce behaves in exactly the same way.

If we denote $\mathcal{D} \subset X(\Omega)$ as the diagonal in $X(\Omega)$; that is, $\mathcal{D} = \{\varphi \in X(\Omega) : \varphi(x) = \text{ a constant}, x \in \Omega\}$ is the set of constant functions, then saying that the system is synchronized is equivalent to having $\mathcal{A} \subset \mathcal{D}$.

If there are parameters λ in the equation, it may happen that the attractor $\mathcal{A}_\lambda \to \mathcal{D}$ as $|\lambda| \to \infty$. In such a case, we say that the system is *almost synchronized* for large $|\lambda|$.

Other types of synchronization problems occur in the applications. For example, it may be that the domain Ω is naturally decomposed into subdomains Ω_j, $j = 1, 2, \ldots, p$, with the diffusive properties through the boundary of these subdomains being much slower than the diffusion on the subdomains. In such a situation, it is natural to investigate the relationship between the dynamics on the subdomains and to understand if they behave in a similar way; that is, almost synchronization of the dynamics on the subdomains.

It can also happen that systems become synchronized in some spatial directions, but not in all directions. This will be the case, for instance, if we are considering domains which are thin in some directions.

For systems of evolutionary equations, some components of the system may undergo much faster diffusion than others and we would expect that some components of the system are synchronized or almost synchronized.

Our objective in this section is to summarize, and interpret in the above context, some known results on the asymptotic behavior of parabolic systems.

2.10.1. Large diffusion in the whole domain – well mixing We begin with the scalar equation

$$
\begin{aligned}
u_t &= d\Delta u + f(u) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{in } \partial\Omega,
\end{aligned}
\tag{2.10.1}
$$

in the space $H^1(\Omega)$, where $d > 0$ is a positive constant.

If the diffusion coefficient is large, the system is said to be well mixed and it reasonable to expect that the asymptotic dynamics is independent of the diffusion; that is, it should be described by the ODE

$$
\dot{u} = f(u). \tag{2.10.2}
$$

A precise statement of this result is the following theorem due to Conway, Hoff and Smoller (1978), Hale (1986).

Theorem 2.10.1. *If f is a C^2-function and there is a $\delta > 0$ such that f*

satisfies the dissipative condition

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} \le -\delta < 0, \tag{2.10.3}$$

then there is a $d_0 > 0$ *such that, for* $d \ge d_0$*, the compact global attractor* $\mathcal{A}_d$ *of (2.10.1) belongs to* $\mathcal{D}$*; that is, the system is synchronized. The attractor* $\mathcal{A}_d = \mathcal{A}_0$ *for all* $d \ge d_0$ *where* $\mathcal{A}_0$ *is the attractor for the ODE (2.10.2).*

PROOF. Condition (2.10.3), the maximum principle and the fact that solutions are regular for $t > 0$ imply that there is a bounded set $B \subset H^1(\Omega)$ such that $\mathcal{A}_d \subset B$ for $d > 0$. If we make the change of variables in (2.10.1), $u = \xi + w$, $\xi = |\Omega|^{-1}\int_\Omega u dx$, then

$$\begin{aligned}
\dot{\xi} \quad &= |\Omega|^{-1}\textstyle\int_\Omega f(\xi + w)dx,\\
w_t \quad &= d\Delta w + f(\xi + w) - |\Omega|^{-1}\textstyle\int_\Omega f(\xi + w)dx\\
&= d\Delta w + h(\xi, w)w.
\end{aligned}$$

On $\mathcal{A}_d$, the function $h(\xi, w)$ is bounded uniformly in $d > 0$. If $0 > -\mu_1 \ge -\mu_2 \ge \dots$ are the eigenvalues of Δ with homogeneous Neumann boundary conditions, then the eigenvalues of $d\Delta|\{w : \int_\Omega w = 0\}$ are given by $-d\mu_j$ and approach $-\infty$ as $d \to \infty$. We can then use this fact to show that there is a $d_0 > 0$ such that $w \to 0$ exponentially as $t \to \infty$ on $\mathcal{A}_d$ if $d \ge d_0$. Since the attractor is invariant, it follows that $w = 0$ on $\mathcal{A}_d$.

REMARK 2.10.1. The same proof holds for the case in which u is an n-vector and d is a diagonal matrix $d = \operatorname{diag}(d_1, \dots, d_n)$ provided that we know that the attractors $\mathcal{A}_d$ are uniformly bounded for $d_j > 0$. If $d_j \ge d_0$, sufficiently large, then $\mathcal{A}_d \subset \mathcal{D}$. To prove the uniform boundedness of $\mathcal{A}_d$ is not trivial in a general setting. However, if there is a sequence of invariant rectangles R_k for the ODE (2.10.2) with $R_k \subset R_{k+1} \to \mathbb{R}^n$ as $k \to \infty$, then it is possible to obtain this uniform boundedness (see, for example, CARVALHO (1995), CARVALHO, CHOLEWA and DLOTKO (1995)).

REMARK 2.10.2. It also is possible to consider the case where u is a vector and the function $f = f(u, \nabla u)$ depends upon the gradient of u provided that the growth rate in ∇u is linear. For large diffusion, the ODE that contains the attractor is $\dot{u} = f(u, 0)$ (see CARVALHO and HALE (1991)).

If we allow spatial dependence in the reaction term $f = f(x, u)$ in (2.10.1), then it is not to be expected that the attractor $\mathcal{A}_d$ is generally in $\mathcal{D}$. However, for large d, we obtain some interesting information. If we introduce the same coordinates as before, we obtain the equations

$$
\begin{aligned}
\dot{\xi} &= |\Omega|^{-1} \textstyle\int_\Omega f(x,\xi+w)dx, \\
w_t &= d\Delta w + f(x,\xi+w) - |\Omega|^{-1} \textstyle\int_\Omega f(x,\xi+w)dx \\
&= d\Delta w + h(x,\xi,w)w + g(x,\xi),
\end{aligned} \tag{2.10.4}
$$

where $g(x,\xi) = f(x,\xi) - |\Omega|^{-1} \int_\Omega f(x,\xi)dx$. If we assume that the attractor $\mathcal{A}_d$ is uniformly bounded for $d > 0$, then $h(x,\xi,w), g(x,\xi)$ is bounded on $\mathcal{A}_d$ for $d > 0$.

We can now apply invariant manifold theory to (2.10.4) to show that $\mathcal{A}_d$ belongs to a smooth graph over $\mathcal{D}$ with representation $w = p(\xi,d)$ if $d \geq d_0$, sufficiently large. Furthermore, there are positive constants C, c such that, for $d \geq d_0$,

$$|p(\xi,d)| \leq \frac{C|\xi|}{d_0 - c}.$$

This implies that $\mathcal{A}_d \to \mathcal{D}$ as $d_0 \to \infty$ and we have almost synchronization.

The flow on $\mathcal{A}_d$ is given by $w(t) = p(\xi(t),d)$, where $\xi(t)$ lies on the attractor for the ODE

$$\dot{\xi} = |\Omega|^{-1} \int_\Omega f(x,\xi + p(\xi,d))dx. \tag{2.10.5}$$

As $d_0 \to \infty$, we obtain the limiting ODE

$$\dot{\xi} = |\Omega|^{-1} \int_\Omega f(x,\xi)dx. \tag{2.10.6}$$

The scalar equation (2.10.6) is gradient and, therefore, if the equilibrium points are hyperbolic, then it is Morse-Smale and the flow for (2.10.5) is topologically equivalent on the attractor to the one for (2.10.6) if d_0 is sufficiently large.

Remark 2.10.3. Similar remarks can be made about systems. Of course, in the remark about topological equivalence, we must include transversal intersection of stable and unstable manifolds if it is a gradient system and Morse-Smale otherwise.

Remark 2.10.4. If we spatiallly discretize our PDE, then we obtain a system of ODE with the components interacting by nearest neighbor coupling. Similar remarks about synchronization and almost synchronization apply to these systems if the strength of the coupling is sufficiently large. For more details, further references and other types of applications and equations, see Hale (1996), Carvalho, Rodriguez and Dlotko (1996)).

2.10.2. Large diffusion in subdomains We begin by considering the 1D equation

$$u_t + (a_\nu(x)u_x)_x = f(u) \quad \text{in } (0,1) \tag{2.10.7}$$

with homogeneous Neumann boundary conditions. If the diffusion coefficient a_ν is assumed to be very large when the parameter ν is small, the same type of argument as in the previous section shows that the dynamics of (2.10.7) is governed by the solutions of the ODE $\dot{\xi} = f(\xi)$ where ξ is the average of u. Therefore, it is reasonable to suppose that, if, for ν small, the diffusion coefficient a_ν is assumed to be very large on a finite number of intervals I_j whose complement in $[0,1]$ is very small, and in most of this complement the diffusion coefficient is very small, then the solutions of (2.10.7) on the attractor should behave essentially as the solutions of a system of ODE for a vector ξ whose j^{th} component ξ_j should correspond approximately to the average of the solution on I_j. The vector field for the j^{th} component should be $f(\xi_j)$ plus some diffusive coupling. These intuitive remarks have recently been made precise by Fusco (1987), Carvalho and Perreira (1994). We now describe their results.

For $\nu \in (0, \nu_0)$, we are going to define a positive diffusion coefficient $a_\nu \in C^2([0,1], \mathbb{R})$. Let $s = (s_0, s_1, \dots, s_n)$, $0 = s_0 < s_1 < \dots < s_n = 1$ be a partition of $[0,1]$ and let $\ell = (\ell_1, \dots, \ell_{n-1})$, $\beta = (\beta_0, \dots, \beta_n)$ be two sequences of positive constants and let $\ell'_1, \dots, \ell'_{n-1}$, $\beta'_0, \dots, \beta'_n$ be functions of ν that approach $\ell_1, \dots, \ell_{n-1}$, $\beta_0, \dots, \beta_n$ from above as $\nu \to 0$. Also, let $\ell_0 = \ell'_0 = 0 = \ell_n = \ell'_n$ and let $e = (e_1, \dots, e_n)$ be another sequence of positive constants. We define $a_\nu = a_\nu(s, \ell, \beta, e)$ in the following manner:

$$
\begin{aligned}
a_\nu(x) &\geq \tfrac{e_i}{\nu} && s_{i-1} + \nu\ell'_{i-1} \leq x \leq s_i - \nu\ell'_i \\
a_\nu(x) &\geq \nu\beta_i && s_i - \nu\ell'_i \leq x \leq s_i + \nu\ell'_i \\
a_\nu(x) &\leq \nu\beta'_i && s_i - \nu\ell_i \leq x \leq s_i + \nu\ell_i
\end{aligned}
\tag{2.10.8}
$$

for $i = 0, 1, \dots, n$.

If we let

$$
\xi_j(t) = \frac{1}{s_j - s_{j-1}} \int_{s_{j-1}}^{s_j} u(x,t)\,dx, \quad j = 1, 2, \dots, n,
$$

where $u(x,t)$ is a solution of (2.10.7) and if we assume that the equilibrium points of (2.10.7) are hyperbolic, then it is shown in Carvalho and Perreira (1994) (a similar result is in Fusco (1987)) that, for ν sufficiently small, the flow on the attractor of (2.10.7) is topologically equivalent to the flow defined by the system of ordinary differential equations

$$
\begin{aligned}
\dot{\xi}_1 &= r_j^2(\xi_2 - \xi_1) + f(\xi_1), \\
\dot{\xi}_j &= r_j^2(\xi_{j+1} - \xi_j) + r_{j-1}^2(\xi_{j-1} - \xi_j) + f(\xi_j), \quad 2 \leq j \leq n-1, \\
\dot{\xi}_n &= r_{n-1}^2(\xi_{n-1} - \xi_n) + f(\xi_n),
\end{aligned}
\tag{2.10.9}
$$

where

$$
r_j^2 = \frac{\beta_j}{2\ell_j(s_j - s_{j-1})}, \quad j = 1, 2, \dots, n.
\tag{2.10.10}
$$

We remark that the method of proof of this result is in the same spirit as the simple one in the previous section. An important thing to prove (which is nontrivial) is that, for any given compact set K in the complex plane, there are at most n eigenvalues in K of the operator $\frac{\partial}{\partial x}(a_\nu \frac{\partial}{\partial x})$ with Neumann boundary conditions if ν is sufficiently small. Furthermore, there are exactly n eigenvalues which approach zero as $\nu \to 0$. Using this fact and invariant manifold theory leads to the above conclusion (see CARVALHO and PERREIRA (1994) for details).

The constants r_j in (2.10.10) depend upon the parameters β_j, ℓ_j and s_j in the definition of the diffusion coefficient a_ν and may assume any real values by changing these parameters. If the constants r_j are chosen to be very large, then the methods of the previous section show that the attractor for the ODE (2.10.9) approaches the diagonal as the $r_j \to \infty$. This implies the dynamics of the PDE on the subdomains behave almost as the average on the subdomain and the *dynamics on the subdomains are almost synchronized* if the diffusion coefficient at the points s_j are chosen in an appropriate way.

Of course, we can also choose the r_j to be very small. The attractor is then very different and the dynamics on the subdomains may be very different.

CARVALHO and CUMINATO (1994) have obtained similar results for a general decomposition of two dimensional domains into subdomains for which the diffusion coefficient is large on each subdomain but small at the boundary. This paper also contains interesting applications to cell tissues.

2.10.3. Large diffusion in some directions – thin domains In this section, we make a few remarks about how results of HALE and RAUGEL (1992) on thin domains are related to the concept of synchronization. We restrict our attention to a thin domain over a line segment. More specifically, we consider the equation

$$u_t - \Delta u = f(u) \quad \text{on } Q_\varepsilon \tag{2.10.11}$$

with Neumann boundary conditions, where $Q_\varepsilon = \{(x,y) : x = \varepsilon g(x), x \in (0,1)\}$, $\varepsilon > 0$ is a small real parameter, $g > 0$ is a C^3-function and f is a C^2-function satisfying some growth conditions which will ensure that we can take the solution space to be $H^1(\Omega)$. If the function f satisfies the dissipative condition (2.10.3), HALE and RAUGEL (1992) have shown that there is a compact global attractor $\mathcal{A}_\varepsilon$ which is bounded uniformly in $\varepsilon > 0$.

To see how this is related to synchronization, let us first suppose that $g(x) \equiv 1$; that is, the domain is the rectangle $Q_\varepsilon = (0,1) \times (0,\varepsilon)$ in $\mathbb{R}^2$. If we scale the rectangle Q_ε to the rectangle $Q = (0,1) \times (0,1)$ by the transformation $x \mapsto x$, $y \mapsto \varepsilon y$, then equation (2.10.11) becomes

$$u_t = u_{xx} + \frac{1}{\varepsilon^2} u_{yy} + f(u) \quad \text{on } Q \tag{2.10.12}$$

with homogeneous Neumann boundary conditions. The space of initial data for (2.10.12) is $H^1_\varepsilon(Q)$, the space $H^1(Q)$ endowed with the norm

$$\|\varphi\|_{H^1_\varepsilon(Q)} = (\|\varphi\|_{H^1(Q)} + \frac{1}{\varepsilon^2}\|\varphi_y\|_{L^2(Q)})^{1/2}.$$

Define the mean value operator

$$M : H^1(Q) \to H^1(0,1), \quad \varphi \mapsto M\varphi = \int_0^1 \varphi(\cdot, y)dy.$$

If we make the transformation $u = v + w$, $v = Mu$, in (2.10.12), we obtain the system

$$\begin{aligned} v_t &= v_{xx} + f(v+w) \quad \text{on } (0,1), \\ w_t &= w_{xx} + \tfrac{1}{e^2}w_{yy} + f(w) - Mf(v+w) \\ &\equiv w_{xx} + \tfrac{1}{\varepsilon^2}w_{yy} + h(v,w)w \quad \text{on } Q, \end{aligned} \tag{2.10.13}$$

with Neumann boundary conditions. The function $h(v,w)$ is uniformly bounded on the attractors $\mathcal{A}_\varepsilon$ for $\varepsilon > 0$.

It is not difficult to show that the eigenvalues of the operator $\partial_x^2 + \varepsilon^{-2}\partial_y^2$ with Neumann boundary conditiions restricted to the space $(I - M)H^1_\varepsilon(Q)$ approach $-\infty$ as $\varepsilon \to 0$. We can now use the same type of argument as in Section 2.10.1 to show that there is an $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, if $w(t)$ belongs to the attractor $\mathcal{A}_\varepsilon$, then $w(t) \to 0$ as $t \to \infty$ exponentially. Since the attractor is invariant, it follows that $w = 0$ on the attractor; that is, elements of the attractor are independent of y. Thus, for $0 < \varepsilon < \varepsilon_0$, the system is *synchronized with respect to* y.

If we allow the function g in the definition of Q_ε to depend upon x and scale to Q by the transformation $x \mapsto x$, $y \mapsto \varepsilon g(x)$, then the resulting differential equations are more complicated. However, using the same mean value operator as above and making the change of variables $u = v + w$, $v = Mu$, Hale and Raugel (1992) show that the proper limit equation on $(0,1)$ is

$$u_t - \frac{1}{g}(gu_x)_x = f(u) \quad \text{in } (0,\, 1). \tag{2.10.14}$$

Let $\tilde{\mathcal{A}}_0 \subset H^1(0,1)$ be the attractor for (2.10.14) and let $\mathcal{A}_0$ be the natural embedding of $\tilde{\mathcal{A}}_0$ into $H^1(Q)$. Hale and Raugel (1992) show that the attractors $\{\tilde{A}_\varepsilon, \varepsilon \geq 0\}$ are upper semicontinuous at $\varepsilon = 0$ in the topology of $H^1_\varepsilon(Q)$. This says that elements of the attractor $\mathcal{A}_\varepsilon$ approach functions which are independent of y as $\varepsilon \to 0$; that is, the system is *almost synchronized in* y.

Since transversality always holds for the 1D equation (2.10.14), it is possible to prove that the flow on $\mathcal{A}_\varepsilon$ is topologically equivalent to the flow on $\mathcal{A}_0$ provided that the equilibrium points of (2.10.12) are hyperbolic. For more details and more general results, see Hale and Raugel (1992).

2.10.4. Large diffusion for some components For applications modeled by systems of reaction-diffusion equations, it often happens that some diffusion coefficients are very large relative to others. In such a situation, it is to be expected that the dynamics for the components of the system with large diffusion can be replaced by an ODE. The limiting system is called a *shadow system* and was first encountered by NISHIURA (1982), FUJII and NISHIURA (1987) in their study of the stability properties of equilibrium solutions of a system of two equations modeling an ecological system. HALE and SAKAMOTO (1988) gave a more complete description involving the attractors of the systems.

To be specific, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and consider the system

$$\begin{aligned} u_t &= d_1 \Delta u + f(u,v) \\ v_t &= d_2 \Delta v + g(u,v) \quad \text{in } \Omega \end{aligned} \tag{2.10.15}$$

with homogeneous Neumann boundary conditions. We take u, v to be scalars, but they could just as well be vectors of different dimensions. We assume that the functions f, g satisfy growth conditions which will permit the discussion of (2.10.15) in the space $H^1(\Omega) \times H^1(\Omega)$.

If we assume that d_1 is fixed and that d_2 is very large, then it is natural to guess that the dynamics of (2.10.15) would approach, as $d_2 \to \infty$, the dynamics of the shadow system

$$\begin{aligned} u_t &= d_1 \Delta u + f(u,z) \\ \dot{z} &= |\Omega|^{-1} \int_\Omega g(u(\cdot,x),z)dx \quad \text{in } \Omega \end{aligned} \tag{2.10.16}$$

with the function u satisfying homogeneous Neumann boundary conditions. If we can show that this is the case, then the dynamics of the v coordinate of the system is essentially independent of the spatial variable; that is, the v *coordinate is almost synchronized*.

To state a precise result, we need some notation. If we let $Z \cong \mathbb{R}$ be the linear space of constant functions in $H^1(\Omega)$, then $H^1(\Omega) = Z \oplus Y$, where $Y = \{w \in H^1(\Omega) : \int_\Omega w(x)dx = 0\}$. For any $v \in H^1(\Omega)$, we can write $v = z + w$, $z = Mv \equiv |\Omega|^{-1} \int_\Omega v(x)dx$, $w \in Y$. The operator M is the mean value operator mentioned in the previous section except applied only to the variable v. We also let $X = H^1(\Omega) \times Z$ and let $N(\delta, A)$ be the δ-neighborhood of a set A in a Banach space and impose the following hypothesis:

(**H**) There is a compact set $K \subset X$ and a constant $\delta_0 > 0$, $d_1^0 > 0$ such that (2.10.16) has a compact attractor $\tilde{\mathcal{A}}_{d_1} \subset K$, $\omega(N(\delta_0, \tilde{\mathcal{A}}_{d_1})) = \tilde{\mathcal{A}}_{d_1}$ for every $d_1 \geq d_1^0$.

If we define $\mathcal{A}_{d_1} = \tilde{\mathcal{A}}_{d_1} \times \{0\}$, $\{0\} \in Y$, then $\mathcal{A}_{d_1}$ is a natural embedding of $\tilde{\mathcal{A}}_{d_1}$ into $H^1(\Omega) \times H^1(\Omega)$. HALE and SAKAMOTO (1988) prove the following result.

THEOREM 2.10.2. *If* (**H**) *is satisfied, then there is a* $d_0^2 > 0$ *such that, if* $d_2 \geq d_2^0$*, then there is a compact attractor* $\mathcal{A}_{d_1,d_2} \in H^1(\Omega) \times H^1(\Omega)$ *for (2.10.15)* *and, for any $\varepsilon > 0$, there is an $\eta > 0$ such that $\mathcal{A}_{d_1,d_2} \subset N(\varepsilon, \mathcal{A}_{d_1})$ if $d_2 \geq \eta$; that is, the sets $\{\mathcal{A}_{d_1}, \mathcal{A}_{d_1,d_2}, d_2 \geq d_2^0\}$ are upper semicontinuous as $d_2 \to \infty$.*

If we assume that the flow on the attractor of (2.10.16) is structurally stable, then it also is possible to prove that the flow for (2.10.15) is topologically equivalent to the flow of (2.10.16) on the attractors.

We remark that a similar result can be stated for compact global attractors if we assume that that these attractors are bounded uniformly with respect to $d_1 \geq d_1^0, d_2 \geq d_2^0$.

The idea for the proof is very similar to the ones that we have mentioned before. We make the transformation $v = z + w$ in (2.10.15) and then use invariant manifold theory to show that the attractor belongs to a smooth manifold which is a graph over $X \times \{0\} \subset H^1(\Omega) \times H^1(\Omega)$ represented by a function $w = h(u, z)$ and $h(u, z) \to 0$ as $d_2 \to \infty$. The details require several estimates which can be found in the above mentioned paper.

3. Autonomous - periodic boundary conditions

3.1. A Poincaré-Bendixson Theorem

In this section, we consider a reaction diffusion equation with periodic boundary conditions or, equivalently on S^1 which we take to be the diffeomorphic image of $[0, 2\pi)$. The objective is to show that a form of the Poincaré-Bendixson Theorem is valid on the ω-limit and α-limit sets of orbits.

3.1.1. Statement of the results We consider the equation

$$u_t = u_{xx} + f(x, u, u_x), \quad x \in S^1, \tag{3.1.1}$$

where $f \in C^2$. For the space of initial data, we choose $X = H^2(S^1)$. By standard methods, one shows that (3.1.1) defines a local semigroup $T(t)$ on X and, if we assume that all solutions are defined for all $t \geq 0$, $T(t)$, $t \geq 0$, is a semigroup on X, which is compact for $t > 0$.

Specific conditions on f which will ensure that these conclusions are valid are the existence of a constant $0 < \gamma < 2$, and a continuous function $C(r)$, $0 \leq r < \infty$, such that, for $x \in S^1$,

$$|f(x, u, p)| \leq C(|u|)(1 + |p|^\gamma) \tag{3.1.2}$$

and the existence of a positive constant K such that

$$uf(x, u, 0) < 0, \quad |u| \geq K. \tag{3.1.3}$$

The condition (3.1.2) assures that solutions will not blow up in finite time and the condition (3.1.3) is a dissipative condition which not only implies that the semigroup $T(t)$ is well defined for $t \geq 0$, but implies that it possesses a global attractor $\mathcal{A}$. The set E of equilibrium points of the equation are the solutions of the equation

$$0 = u_{xx} + f(x, u, u_x), \quad x \in S^1, \tag{3.1.4}$$

For separated boundary conditions, we have seen in Section 2.1 that the ω-limit set of any bounded orbit is a singleton. For periodic boundary conditions, there may be other minimal sets. In fact, it is possible to have a closed orbit which is generated by a solution which is periodic in time. For example, the linear equation

$$u_t = u_{xx} + u_x + u, \quad x \in S^1, \tag{3.1.5}$$

has the solution $\cos(t + x)$ of period 2π. This solution also is called a *rotating wave* of wave speed 1.

The following general result is due to FIEDLER and MALLET-PARET (1989).

THEOREM 3.1.1. *If $\gamma^+(u_0)$ is bounded, then $\omega(u_0)$ satisfies exactly one of the following alternatives:*

(i) $\omega(u_0)$ is a periodic orbit,
(ii) $\alpha(v_0) \subset E$ and $\omega(v_0) \subset E$ for each $v_0 \in \omega(u_0)$.

In particular, the Poincaré-Bendixson property holds on the ω-limit sets of trajectories in the sense that it is either a periodic orbit or consists of homoclinic or heteroclinic orbits connecting equilibrium points.

For $f = f(u, u_x)$ independent of x, we say that a solution u of (3.1.1) is a *rotating wave of wave speed $c \neq 0$* if it can be represented in the form $\varphi(x + ct)$. It is clear that any rotating wave is a periodic solution of (3.1.1). For f independent of x, MASSATT (1986) proved a version of Theorem 3.1.1 which stated that, for any $u_0 \in X$, either $\omega(u_0)$ is a single periodic orbit or a set of equilibria which differ only by shifting x (*a standing wave*). The same result was proved by MATANO (1988), who further showed that $\omega(u_0)$ is a single equilibrium if $f(u, u_x) = f(u, -u_x)$. Independently, assuming that $f(u, u_x)$ is analytic, ANGENENT and FIEDLER (1988) proved a weaker version of Theorem 3.1.1 which stated that, if $v_0 \in \omega(u_0)$, then $\omega(v_0)$ and $\alpha(v_0)$ contains a periodic orbit or an equilibrium point. Under these same hypotheses, they also showed that every periodic orbit of (3.1.1) is a rotating wave.

If $f = f(u, u_x)$ is independent of x, then we can transform (3.1.1) to rotating coordinates $w(t, x) = u(t, x + ct)$ to obtain the equation

$$w_t = w_{xx} + f(w, w_x) + cw_x, \quad x \in S^1. \tag{3.1.6}$$

In this way, stationary solutions become rotating waves (for $c \neq 0$). Applying Theorem 3.1.1 to (3.1.6) yields the result of MASSATT (1986) that $\omega(u_0)$ is a single periodic orbit if it contains any periodic solution at all. Under the additional symmetry condition of MATANO (1988), there can be no periodic solutions of (3.1.1) and therefore $\omega(u_0)$ is an equilibrium solution up to an x shift. By additional symmetry arguments, MATANO (1988) shows that it is a single equilibrium point.

The similarity between (3.1.1) and planar systems can be made more precise in the following way. For any $x_0 \in S^1$, define the projection

$$\begin{aligned} \pi_{x_0} \quad & : X \to \mathbb{R}^2 \\ \varphi \mapsto \pi_{x_0}(\varphi) \quad & = (\varphi(x_0), \varphi_x(x_0)). \end{aligned}$$

THEOREM 3.1.2. *If $\gamma^+(u_0)$ is bounded, then $\pi_{x_0} : \omega(u_0) \to \mathbb{R}^2$ is a homeomorphism onto the compact subest $\pi_{x_0}\omega(u_0)$ of $\mathbb{R}^2$.*

For any $x_0, x_1 \in S^1$, the sets $\pi_{x_0}\omega(u_0)$ and $\pi_{x_1}\omega(u_0)$ are homeomorphic by the homeomorphism $\pi_{x_1}\pi_{x_0}^{-1}$. In fact, the induced flows on $\pi_{x_0}\omega(u_0)$ and $\pi_{x_1}\omega(u_0)$ are topologically conjugate through this homeomorphism.

We emphasize that Theorem 3.1.2 applies only to the ω-limit sets of an orbit and not to the global attractor if it exists. In fact, the global attractor contains all of the unstable sets of equlibrium points and periodic orbits and is, in general, not a two dimensional object.

It is not even true that an orbit whose limit set is a periodic orbit is injective by the projection π. In fact, (3.1.5) has the solution

$$u(t,x) = \cos(t+x) + e^{-3t}\cos 2(t+x)$$

and, for $x_0 = 0$,

$$\pi_0 u(t,\cdot) = (\cos t, -\sin t) + e^{-3t}(\cos 2t, -2\sin 2t).$$

Obviously, $\pi_0 u(t,\cdot)$ intersects itself as well as its ω-limit set at arbitrarily large times.

3.1.2. An abstract version for semigroups FIEDLER and MALLET-PARET (1989) have given axioms for an abstract class of semigroups for which Theorems 3.1.1 and 3.1.2 are true. In addition to being applicable to other types of systems, the proofs in the abstract setting make the arguments more transparent and simpler.

Suppose that $T(t) : X \to X$, $t \geq 0$, is a C^0-semigroup with the property that $T(t)$ is compact for $t > 0$. For simplicity in notation, we let $u(t) = u(t, u_0) = T(t)u_0$ or, sometimes, if another symbol is needed, $v(t) = T(t)v_0$, etc. and refer to these functions as solutions. If $u(t, u_0)$ is a solution, we suppose that, for $t > 0$, $u_t(t, u_0)$ exists and depends continuously on t, u_0. If $v_0 \in \omega(u_0)$, then we know that negative orbits exist through v_0 but we do not assume that they are unique. Therefore, when we speak of $\alpha(v_0)$, we understand that we are speaking of the α-limit set of any negative orbit through v_0.

We suppose that there exist maps

$$z : X \to \{0, 1, 2, \dots, \infty\}$$

$$\pi : X \to \mathbb{R}^2, \text{ linear}$$

such that the following axioms hold for any solutions $u_1(t)$, $u_2(t)$:

(A0) (*Finiteness*) $z(u^1(t) - u^2(t))$ is finite for $t > 0$.
(A1) (*z-dropping*) If $t_0 > 0$ and $\pi(u^1(t_0) - u^2(t_0)) = 0$, then either $u^1(t_0) - u^2(t_0) = 0$ or the map $t \mapsto z(u^1(t) - u^2(t))$ drops strictly at t_0; that is, for any $\varepsilon > 0$, $z(u^1(t_0 + \varepsilon) - u^2(t_0 + \varepsilon)) < z(u^1(t_0 - \varepsilon) - u^2(t_0 - \varepsilon))$.
(A2) (*Continuity*) If the map $t \mapsto z(u^1(t) - u^2(t))$ does not drop strictly at t_0 and if $u^1(t_0) - u^2(t_0) \neq 0$, then z is locally constant; that is, there exists a neighborhood u of $u^1(t_0) - u^2(t_0)$ in X such that, for any $\varphi \in U$, $z(\varphi) = z(u^1(t_0) - u^2(t_0))$.
(A3) (*Regularity*) Axioms (A0)-(A2) hold with u_t replacing $u^1(t) - u^2(t)$, where $u(t) = T(t)u_0$ for any $u_0 \in X$.

If Axioms (A0)-(A4) are satisfied, Fiedler and Mallet-Paret (1989) show that Theorems 3.1.1 and 3.1.2 are true.

For the scalar parabolic equation (3.1.1), the map π is the one defined above and the map z is the zero number of a function; that is, $z(\varphi)$ is the number of zeros of the function $\varphi(x)$ in the interval $[0, 2\pi)$. We have remarked before that $z(u(t))$ is a nonincreasing function of t for any solution of (3.1.1). The fact that there is strict dropping of $z(u(t))$ at t_0 if there is a multiple zero is a consequence of Angenent (1988). The fact that the axioms are satisfied for the differences of two solutions as well as the derivative also is a consequence of Angenent (1988) if we study the corresponding linear equations for which these functions are solutions.

Certain types of cyclic systems of ODE also satisfy these axioms. The following cyclic system was considered by Mallet-Paret and Smith (1990):

$$\dot{u}_j = f_j(u_j, u_{j-1}), \quad j \pmod{n}, \tag{3.1.7}$$

where each function $f_j : \mathbb{R}^2 \to \mathbb{R}$ is C^1 and there are constants $\delta_j \in \{-1, +1\}$ such that $\prod_{j=1}^{n} \delta_j = 1$ and

$$\delta_j D_2 f_j(\xi, \eta) > 0, \quad \text{for all } \xi, \eta, j, \tag{3.1.8}$$

where D_2 represents differentiation with respect to the second argument. The integer valued map z associated with this system is defined by $z(0) = 0$ and, for $u = (u_1, \dots, u_n) \neq 0$,

$$z(u) = \operatorname{card}\{\ j : \exists k \geq 1 \text{ with } u_j u_{j-k} \neq 0, \\ u_{j-i} = 0,\ 1 \leq i \leq k-1,\ \textstyle\prod_{i=j}^{j-k+1} \delta j - i < 0\}.$$

In case $u_j \neq 0$ for all j, then

$$z(u) = \text{ card}\{j : \delta_j u_j u_{j-1} < 0\}.$$

The map $\pi : \mathbb{R}^n \to \mathbb{R}^2$ is defined by $\pi u = (u_{j-1}, u_j)$ for some fixed j.

It is possible to show that (A0)–(A3) hold for (3.1.7) with this definition of z, π.

Another interesting class of ODE has been considered by FUSCO and OLIVA (1990)). Let e be a given basis for $\mathbb{R}^n$. We let $\mathcal{L}(e)$ be the subset of of the bounded linear operators on $\mathbb{R}^n$ with the property that, for any $L \in \mathcal{L}(e)$, the matrix representation of L in the basis e has the form

$$L = \begin{bmatrix} a_1 & b_1 & 0 & \dots & 0 & c_1 \\ c_2 & a_2 & b_2 & \dots & 0 & 0 \\ 0 & c_3 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} & b_{n-1} \\ b_n & 0 & 0 & \dots & c_n & a_n \end{bmatrix} \tag{3.1.9}$$

where

$$b_j \geq 0, c_j \geq 0, 1 \leq j \leq n, \quad \prod_{j=1}^{n} b_j + \pi_{j=1}^{n} c_j > 0. \tag{3.1.10}$$

With this notation, FUSCO and OLIVA (1990) consider the set of ODE

$$\dot{u} = f(u), \quad Df(u) \in \mathcal{L}(e). \tag{3.1.11}$$

Any ODE of the form (3.1.11) is a special case of a cooperative system (a cooperative system only requires that the off diagonal terms of the Jacobian matrix are non-negative). These systems are monotone (HIRSCH (1983), (1988)).

The system (3.1.11) includes systems obtained by spatial discretization of (3.1.1). It also includes some of the cyclic systems considered above; namely, those for which each $\delta_j = 1$, $1 \leq j \leq n$.

If we define the maps z and π as before, then it is possible to show that (A0)–(A3) are satisfied. Therefore, Theorems 3.1.1 and 3.1.2 are satisfied.

For system (3.1.11), FUSCO and OLIVA (1990) proved in addition that the stable and unstable manifolds of two hyperbolic periodic orbits always are transversal (another analogy with the Poincaré-Bendixson Theorem). It is feasible that the same result is true for (3.1.1).

3.2. Index and connecting orbits

In this section, we consider (3.1.1) with the function f independent of x and analytic in u, u_x; that is, the equation

$$u_t = u_{xx} + f(u, u_x), \quad x \in S^1, \tag{3.2.1}$$

where f is analytic.

From the previous section, the only miminal sets of (3.2.1) are either periodic orbits or equilibrium points. Furthermore, every periodic orbit is either a rotating wave or a standing wave. If we assume that (3.1.2), (3.1.3) are satisfied, then there is a compact global attractor $\mathcal{A}$ for (3.2.1). If we let $\mathcal{M}$ be the set consisting of all of the equilibrium points, rotating waves and standing waves and assume that each element ψ of $\mathcal{M}$ is hyperbolic, then

$$\mathcal{A} = \cup_{\psi \in \mathcal{M}} W^u(\psi),$$

where $W^u(\psi)$ is the unstable manifold of ψ.

For the system (3.2.1), Angenent and Fiedler (1988) give some more detailed information about the behavior of the solutions on the global attractor $\mathcal{A}$ including the index of rotating waves and connecting orbits. We describe briefly their results.

3.2.1. Index of rotating waves Recall that the zero number $z(\varphi)$ of $\varphi \in X$ is the number of sign changes of φ, not counting multiplicity; that is, $z(\varphi)$ is the maximal integer n such that there exist $0 \le x_{n+1} = x_0 < x_1 < \cdots < x_n < 2\pi$ with $\varphi(x_i)\varphi(x_{i+1} < 0$, $0 \le i \le n$.

The discussion of the index of a minimal set of (3.2.1) involves knowledge of the spectrum of the linear variational operator about an orbit describing this set. Since these minimal sets are either rotating waves or equilibrium points, this involves the consideration of a linear equation of the form

$$v_t = v_{xx} + a(t,x)v_x + b(t,x)v, \quad x \in S^1. \tag{3.2.2}$$

If the minimal set is an equilibrium point, then the coefficients are independent of t. If the minimal set corresponds to a rotating wave, then the coefficients are periodic in t of some period ω. In any case, the index of the minimal set is determined by an investigation of the spectrum of the Poincaré map $T : X \to X$, $v_0 \mapsto v(\cdot, \omega, v_0)$ for some $\omega > 0$, where $v(x,t,v_0)$ is the solution of (3.2.2) with initial value v_0 at $t = 0$.

The map T is compact and the spectrum of T consists of the element 0 and eigenvalues $\lambda_j, j = 0, 1, 2, \ldots,$ of finite multiplicity which we can order as $|\lambda_j| \ge |\lambda_{j+1}|$ and they are repeated in the sequence according to their multiplicity.

Angenent and Fiedler (1988) prove the following results.

Theorem 3.2.1. *For all $j \ge 0$, $|\lambda_{2j}| > |\lambda_{2j+1}|$. Therefore, λ_0 is an isolated real simple eigenvalue with eigenspace E_0 of dimension 1 and, for each $j \ge 1$, $\{\lambda_{2j-1}, \lambda_{2j}\}$, is a spectral set with real generalized eigenspace E_{2j} of dimension 2.*

Theorem 3.2.2. *Any nonzero $u \in E_{2j}$ has only simple zeros and $z(u) = 2j$.*

The proof of these results uses the fact that the zero number of a solution is nonincreasing in t to first prove that the zero number is finite for any function in the generalized eigenspace of those eigenvalues which have the same modulus. The next step is to observe that the theorems are true if $a = 0 = b$ and then to use a homotopy argument replacing (a, b) by $(\theta a, \theta b)$, $0 \le \theta \le 1$ to obtain the general result. This is accomplished by showing that the zero number is continuous in θ and therefore must remain constant.

COROLLARY 3.2.1. *For any rotating or standing wave φ of (3.2.1),*

$$i(\varphi) \in [z(\varphi_x) - 1, z(\varphi_x)].$$

In particular, each such wave is unstable.

The instability of rotating waves follow from more general results on monotone systems. In fact, it is known that, for almost all initial data, the ω-limit set is a singleton and thus an equilibrium point. Therefore, all periodic orbits of (1.1) are unstable (see, for example, HIRSCH (1983)).

3.2.2. Connecting orbits To state a precise result, suppose that $\psi \in \mathcal{M}$ is hyperbolic with $i(\psi) > 0$ and define $w_\pm \in X$ by the relations

$$\begin{aligned} w_+ &= \inf\{w > \psi : w \text{ is an equilibrium point}\} \\ w_- &= \inf\{w < \psi : w \text{ is an equilibrium point}\} \end{aligned} \tag{3.2.4}$$

Then it can be shown that w_+ and w_- are well defined, are homogeneous steady states with index zero (stable), $w_- < \psi < w_+$, and there is an orbit connecting ψ to w_+ and an orbit connecting ψ to w_-.

The following result gives information about some of the connecting orbits.

THEOREM 3.2.3. *Suppose that f in (3.2.1) is analytic and each element of $\mathcal{M}$ is hyperbolic. For $\psi \in \mathcal{M}$, $i(\psi) > 0$, define $w_\pm$ by (3.2.4). If k is an integer $0 < 2k \le i(\psi)$, then there exists a $w_k \in \mathcal{M}$ such that $w_- < w_k < w_+$, $z(w_k - \psi) = 2k$, and there is an orbit connecting ψ to w_k.*

3.2.3. An example Let us consider a specific example from Angenent and Fiedler (1988) for which there exist many rotating waves. Consider the equation

$$u_t = \varepsilon u_{xx} + f(u)_x + g(u), \quad x \in S^1, \tag{3.2.5}$$

where $\varepsilon > 0$ is a parameter,

$$f(u) = \frac{1}{3}u^3, \quad g(u) = u(1 - (\delta u)^2), \tag{3.2.6}$$

where $\delta > 0$ is a small constant. The term $\delta^2 u^3$ in h is introduced only to make the system dissipative and will not play a role in the following construction of rotating waves.

A rotating wave φ of (3.2.5) with speed c is a periodic solution of the equation

$$\varepsilon\varphi_{xx} + [(f(\varphi) + c\varphi]_x + g(\varphi) = 0$$

or the equivalent system

$$\begin{aligned} \varepsilon\varphi_x &= \psi - f_c(\varphi) \\ \psi_x &= -g(\varphi), \end{aligned} \tag{3.2.7}$$

where $f_c(u) = f(u) + cu$.

If we set $\delta = 0$, equation (3.2.7) is the classical van der Pol equation. For $c < 0$, there is a unique periodic solution of this equation for each $\varepsilon > 0$. If $\varepsilon > 0$ is small, this periodic solution is a relaxation oscillation following the curve $\psi = f_c(\varphi)$ most of the time, except for two rapid transition layers of width $O(\varepsilon^{2/3})$. The minimal period of the solution is given by

$$p_\varepsilon = -(3 - 2\log 2)c + O(\varepsilon^{2/3}) \tag{3.2.8}$$

and the amplitude is approximately $2c^{2/3}$. For δ small, these solutions will still exist and correspond to rotating waves for equation (3.2.5). There will be a finite number of rotating waves with wave speed

$$c_m = -\frac{2\pi}{3 - 2\log 2} \cdot \frac{1}{m} + o(1),$$

where m is an integer satisfying $1 < m \le M_\varepsilon$, where $M_\varepsilon \to \infty$ as $\varepsilon \to 0$. The upper bound occurs because of the expression for the period p_ε in (3.2.8). The corresponding rotating wave φ_m has $2m$ rapid transition layers, $z(\varphi_m) = 2m$. Also, $i(\varphi_m) = 2m - 1$ by Corollary 3.2.1. Because of the uniqueness of the periodic orbit of (3.2.7), these are the only rotating waves that can occur in (3.2.6). As a consequence of Theorem 3.2.3, there is a connecting orbit between φ_m and each wave φ_j with $j < m$ and a connecting orbit between φ_m and each of the equilibrium points $+1/\delta$ and $-1/\delta$. This illustrates some of the complexity of the flow on the attractor. In particular, we see that a periodic orbit can be heteroclinic to more than two other periodic orbits.

3.3. Small viscosity limit of an inhomogeneous conservation law

In this section, we consider a differential equation for a conservation law with a source term

$$u_t + f(u)_x = g(u), \quad x \in S^1, \tag{3.3.1}$$

as well as the equation with a small viscosity term

$$u_t - \varepsilon u_{xx} + f(u)_x = g(u), \quad x \in S^1. \tag{3.3.2}$$

We assume that the functions f, g satisfy the following hypotheses:

(H$_1$) $f \in C^2(\mathbb{R}, \mathbb{R})$, $f''(u) > f_0 > 0$.
(H$_2$) $g(u)$ has finitely many zeroes, $a_1 < a_2 < \ldots < a_{2k+1}$, all simple; and there is a constant $M_0 > 0$ such that

$$ug(u) < 0 \text{ for } |u| > M_0. \tag{3.3.3}$$

Remark 3.3.1. We emphasize that the function f in (3.3.1), (3.3.2) is strictly convex and so we are not discussing the same equation as in (3.3.3) in spite of the apparent similarity. The function $f(u)$ in (3.3.3) is a cubic and therefore is not convex. It is an interesting problem to see to what extent the results to be given below hold without the assumption of convexity. Partial results are in Lyberopoulos (1990).

3.3.1. Dynamics of inhomogeneous conservation laws For (3.3.1), the initial data u_0 is assumed to be in the space

$$BV(S^1) = \{\, u \in L^1(S^1) : \, u_x \text{ is a finite measure on } S^1 \,\}$$

of functions of bounded variation on S^1. Under the hypothesis (H$_1$), a solution $u(x,\, t)$ of (3.3.1) is said to be an *admissible solution* if, for almost all $t > 0$, the right and left hand limits of $u(x,\, t)$ with respect to x exist and satisfy the entropy condition

$$u(x^-,\, t) \geq u(x^+,\, t)\,. \tag{3.3.4}$$

It is known that the initial value problem has a unique weak admissible solution $u(x,\, t)$ with the property that $u \in C^0((0,\, \infty) : L^1(S^1))$, $u(\cdot,\, t) \in \mathrm{BV}(S^1)$ (see, for example, Kruzkov (1970), Vol'pert (1967)). We consider only admissible solutions and, for any $u_0 \in BV(S^1)$, we let $u^0(\cdot, t, u_0) \equiv \Phi^0(t)u_0$ be the solution of (3.3.1) with initial data u_0.

Since $\mathrm{BV}(S^1) \subset L^p(S^1)$ for any $0 < p < +\infty$, we can discuss the flow defined by (3.3.1) in $L^p(S^1) \cap \mathrm{BV}(S^1)$. We consider convergence of this flow in the topology of $L^p(S^1)$ and emphasize this by writing the ω-limit set of a subset B of $\mathrm{BV}(S^1)$ as $\omega_{L^p}(B) = \cap_{\tau \geq 0} \mathrm{Cl}_{L^p} \cup_{t \geq \tau} \Phi^0(t)B$. We define α-limit sets α_{L^p}, attractors, etc. in a similar way. Since $BV(S^1)$ is compactly embedded in $L^p(S^1)$, $0 < p < \infty$, it follows that the set $\{\cup_{t \geq 0} \Phi^0(t)B\}$ being bounded implies that it is relatively compact in $L^p(S^1)$ and, thus, $\omega_{L^p}(B)$ is compact and invariant. The same remark applies to α-limit sets.

We say that $\psi : \mathbb{R} \to \mathbb{R}$ is a *traveling wave solution of (3.3.1) with wave speed* c if $\psi(x - ct)$ is a solution of (3.3.1) for all $(x, t) \in S^1 \times \mathbb{R}$. We say that $\omega_{L^p}(u_0)$ *is a traveling wave with wave speed* c of (3.3.1) if there is a real constant c and a traveling wave solution $\psi : \mathbb{R} \to \mathbb{R}$ of (3.3.1) such that, for any $\varphi \in \omega_{L^p}(u_0)$, there is a constant d_φ such that $u^0(x,\, t,\, \varphi) = \psi(x - ct + d_\varphi)$.

The following result on ω-limit sets for orbits of (3.3.1) has been proved by Fan and Hale (1993), Lyberopoulos (1994).

Theorem 3.3.1. *If* (H_1), (H_2) *are satisfied and* $u(x,\, t)$ *is an admissible solution of (3.3.1), then, for any* $0 < p < +\infty$, $\omega_{L^p}(u(\cdot,\, 0))$ *exists and is either one of the zeros of* g *or a traveling wave with wave speed* $f'(a_{2m})$, *where* $m = m(u_0)$ *depends upon* $u(\cdot,\, 0) \equiv u_0$.

For the globally defined and bounded solutions of (3.3.1), Fan and Hale (1995) obtained the following result. Theorem 3.3.2. *Let* $\mathcal{A}_0$ *be the set of* $\varphi \in BV(S^1)$ *such that, the solution* $\Phi^0(t)\varphi$ *of (3.3.1) is defined and bounded in* $BV(S^1)$ *for* $t \in \mathbb{R}$. *Then, for any* $0 < p < +\infty$, *and any* $\varphi \in \mathcal{A}_0$, $\alpha_{L^p}(\varphi)$ *and* $\omega_{L^p}(\varphi)$ *exist and are either an equilibrium point or a traveling wave. Furthermore,* $\mathcal{A}_0$ *is a global attractor in* $L^p(S^1)$; *that is,* $\mathcal{A}_0$ *is invariant, and, for any bounded set* $B \subset BV(S^1)$,

$$\mathrm{dist}_{L^p(S^1)}(\Phi^0(t)B, \mathcal{A}_0) \to 0 \text{ as } t \to \infty.$$

Fan and Hale (1995) show that it is possible to give a more complete characterization of relationships between the α-limit set and ω-limit set of an orbit. To describe the result, we need some additional notation. If $u_c(x,t)$ is a solution of (3.3.1) which exists for $t \in \mathbb{R}$ and $\varphi(x-ct)$ is a traveling wave such that $u_c(x,t) - \varphi(x-ct) \to 0$ as $t \to \infty$, then we say that $u_c(x,t)$ *connects to* φ *at* $t = +\infty$. Similarly, if there exists a traveling wave $\psi(x-ct)$ such that $u_c(x,t) - \psi(x-ct) \to 0$ as $t \to -\infty$,then we say that $u_c(x,t)$ connects to ψ at $t = -\infty$. For any function φ on $S^1 \cong [0, 2\pi)$, we define the *oscillation number* Card $\mathcal{N}_\varphi$ as the cardinal number of the set

$$\mathcal{N}_\varphi = \{\, x \in S^1 : \varphi(x^+) = \varphi(x^-) = 0\,\}.$$

Theorem 3.3.3.

(i) *Any element of* $\mathcal{A}_0$ *must connect at* $t = +\infty$ *to either a travelling wave or a constant* $\in \{a_1, \ldots, a_{2k+1}\}$.

(ii) *If* $u_c(x,t)$ *is not a traveling wave and connects to a nonconstant traveling wave* φ *of (3.3.1) at* $t = +\infty$ *with speed* $f'(a_{2m})$ *for some* $m \in \{1, \ldots, k\}$, *then* $u_c(x,t)$ *must connect to either a traveling wave* ψ *of (3.3.1) at* $t = -\infty$ *with the same speed or* φ *must connect to a constant solution* $\psi \equiv a_{2m}$ *at* $t = -\infty$. *Furthermore, Card* $\mathcal{N}_{\varphi - a_{2m}} \le$ *Card* $\mathcal{N}_{\psi - a_{2m}}$.

(iii) *If* $u_c(x,t)$ *connects to a constant* b *at* $t = +\infty$ *and* $u_c(x,t) \neq b$ *for all* x, t, *then* $b = a_{2m+1}$ *for some* m *and* $u_c(x,t)$ *connects at* $t = -\infty$ *to a traveling wave* ψ *with speed either* $f'(a_{2m})$ *or* $f'(a_{2m+2})$.

3.3.2. Viscosity limit of the attractors In this subsection, we discuss some relationships between the flow defined by (3.3.1) and the flow defined by (3.3.2). The space of initial data for (3.3.2) is taken to be the set of functions in $H^2(S^1)$.

For any $u_0 \in H^2(S^1)$, we let $u^\varepsilon(\cdot, t, u_0) \equiv \Phi^\varepsilon(t)u_0$ denote the solution of (3.3.2) with initial data u_0. The mapping $\Phi^\varepsilon(t), t \geq 0$ is a C^0-semigroup on $H^2(S^1)$ and is compact for $t > 0$.

For the semigroup $\Phi^\varepsilon(t)$ on $H^2(S^1)$, we can define the *positive orbit* $\gamma^+(B)$ of a subset B of $H^2(S^1)$, the ω-*limit set* $\omega_{H^2}(B)$ of B, invariant sets and the global attractor for the semigroup $\Phi^\varepsilon(t)$. As we have noted above, the global attractor $\mathcal{A}_\varepsilon$ exists for (3.3.2). Furthermore, for any $\varphi \in \mathcal{A}_\varepsilon$, either $\omega_{H^2}(\varphi)$ is a traveling wave or $\alpha_{H^2}(\varphi)$ and $\omega_{H^2}(\varphi)$ belongs to the set of equilibrium points of (3.3.2) (see Theorem 3.1.1).

The ultimate goal would be to prove that the limit of $\mathcal{A}_\varepsilon$ as $\varepsilon \to 0$ is equal to $\mathcal{A}_0$. At this time, only partial results have been achieved. More precisely, Fan and Hale (1995) have proved the following result.

Theorem 3.3.4. *Assume the hypotheses* ($\mathbf{H}_1$), ($\mathbf{H}_2$).

(i) *For any set $B = \{u^\varepsilon(x,t), 0 < \varepsilon \leq \varepsilon_0\}$ of connecting orbits of (3.3.2), there is a subsequence $\{u^{\varepsilon_n}(x,t)\}$ of B converging to $u(x,t)$ a.e. in $S^1 \times \mathbb{R}$ as $n \to \infty$ and $u(x,t)$ is a connecting orbit of (3.3.1).*

(ii) *For any set $\{\varphi^\varepsilon(x - c_\varepsilon t), 0 < \varepsilon \leq \varepsilon_0\}$ of traveling wave solutions of (3.3.2), there is a subsequence $\{\varphi^{\varepsilon_n}(x - c_{\varepsilon_n} t)\}$ such that*

$$\varphi^{\varepsilon_n}(x - c_{\varepsilon_n} t) - \varphi(x - ct) \to 0$$

a.e. in $S^1 \times \mathbb{R}$ as $n \to \infty$, where φ is a traveling wave solution of (3.3.1).

We remark that dim $\mathcal{A}_0 = +\infty$, whereas dim $\mathcal{A}_\varepsilon < +\infty$ for each $\varepsilon > 0$. Therefore, it is very interesting to know if $\mathcal{A}_\varepsilon \to \mathcal{A}_0$ as $\varepsilon \to 0$ in the Hausdorff sense. The above results almost show this fact. It only remains to show that every element of $\mathcal{A}_0$ (which can have traveling waves which are discontinuous) can be approximated by elements of $\mathcal{A}_\varepsilon$ if ε is small. The boundary layer problem in this discussion seems to be very interesting but nontrivial.

3.4. Nonlocal reaction terms

We have seen in Section 3.2 that each rotating wave of a reaction diffusion equation on S^1 is unstable. In this section, we show that it is possible to have a stable rotating wave if we allow the reaction term to depend upon a nonlocal term. We recall that such situations can arise if we consider systems of two equations for which one of the diffusion coefficients is large compare with the other; that is, shadow systems. In analogy with the discussion in Section 2.8, it is only necessary to show that it is possible to obtain a primary Hopf bifurcation from a stable equilibrium when we include nonlocal terms. As a consequence, we need only to point out how the eigenvalues of a linear equation depend upon these nonlocal terms.

We consider only the simplest eigenvalue problem

$$v_{xx} + \alpha v + \beta \frac{1}{2\pi} \int_0^{2\pi} v(y)\,dy + \gamma v_x = \mu v, \quad x \in S^1 \simeq [0, 2\pi), \tag{3.4.1}$$

where α, β, γ are constants, $\gamma > 0$, and μ is the eigenvalue parameter. Since v is 2π-periodic, we can insert the Fourier series for v to obtain the eigenvalues

$$\mu_0 = \alpha + \beta, \quad \mu_n^{\pm} = -n^2 + \alpha \pm i\gamma, \ n \geq 1,$$

with each eigenvalue being simple with a basis for the eigenvectors corresponding to μ_0 being the constant function 1 and a real basis for the span of the eigenvalues $\{\mu_n^-, \mu_n^+\}$ being $\{\sin nx, \cos nx\}$.

If (3.4.1) corresponds to the linear variational equation about a constant solution of a nonlinear equation, then we are interested in the point in the α, β plane at which the first bifurcation can occur; that is, the points α, β for which there are eigenvalues on the imaginary axis and all other eigenvalues have real parts < 0. Since $\operatorname{Re} \mu_n = -n^2 + \alpha$ for $n \geq 1$, we have $\operatorname{Re} \mu_1 > \operatorname{Re} \mu_n$ for $n \geq 2$. Therefore, we are interested in those values of α, β on the sets $\Omega_0 = \{\alpha, \beta : \alpha = -\beta, \alpha \geq 1\}$ and $\Omega_1 = \{\alpha = 1, -\infty < \beta \leq -1\}$. On Ω_1, there is Hopf bifurcation which leads to a stable traveling wave if it is supercritical. On Ω_0, there is a bifurcation to a stable equilibria if it is supercritical.

It is easy to construct examples for which such bifurcations will be a supercritical Hopf and thus obtain a stable rotating wave.

4. Nonautonomous equations

4.1. Periodic - separated boundary conditions

In this section, we consider the equation

$$u_t = u_{xx} + f(t, x, u, u_x) \quad \text{in } \Omega = (0, 1), \tag{4.1.1}$$

with the boundary conditions as in Section 2.1,

$$\alpha u_x + \beta u|_{x=0} = 0 = \gamma u_x + \delta u_x|_{x=1}, \tag{4.1.2}$$

where $\alpha, \beta, \gamma, \delta$ are constants which can be normalized so that $\alpha^2 + \beta^2 = 1 = \gamma^2 + \delta^2$. The function f is C^2 and periodic in t of period ω.

We assume that there is a Banach space X (usually a space of functions $H^s(0, 1)$ for s respecting the boundary conditions (4.1.2)) such that the initial value problem of (4.1.1), (4.1.2), $u|_{t=\tau} = \varphi \in X$ has a unique solution $T(t, \tau)\varphi \in X$ which is defined for all $t \geq \tau$, continuous in all variables and is C^1 in φ. Also, we suppose that, if B is a bounded set in X, then the closure of $T(t, \tau)B$ is compact if $t > \tau$.

The *Poincaré map* $\pi : X \to X$ of (4.1.1), (4.1.2) is defined to be $\pi\varphi = T(\omega, 0)\varphi$, $\varphi \in X$. Fixed points of π are in one-to-one correspondence with the solutions of (4.1.1), (4.1.2) which are ω-periodic in t. We denote the fixed point set of π by $\operatorname{Fix}(\pi)$.

If $\varphi \in X$, we let $\gamma_\pi^+(\varphi) = \{\pi^n\varphi, n \geq 0\}$ be the *positive orbit* of φ under the action of π and $\omega_\pi(\varphi)$ be the *ω-limit set of φ under* π. Similarly, we define the *negative orbit* $\gamma_\pi^-(\varphi)$ and the *α-limit set* $\alpha_\pi(\varphi)$.

The following result is due to Chen and Matano (1989) for $f = f(t, u)$ and to Brunovsky, Poláčik and Sanstede (1992) for the general case. The proof depends heavily upon the nonincrease of the zero number $z(u)$ for solutions of linear 1D parabolic nonautonomous equations.

Theorem 4.1.1. *If $\gamma_\pi^+(\varphi)$ (resp. $\gamma_\pi^+(\varphi)$) is bounded, then $\omega_\pi(\varphi)$ (resp. $\alpha_\pi(\varphi)$) is a singleton in Fix(π).*

Remark 4.1.1. If we assume that the function f in (4.1.1) is independent of t, then we have seen in Section 2.1 that the system is a gradient system and the ω-limit set of any bounded orbit is an equilibrium point. Theorem 4.1.1 says that each bounded solution of the 1D equation (4.1.1) approaches a periodic solution of the same period as f; that is, a harmonic solution.

Remark 4.1.2. As we have noted in Chapter 1, if $\pi : X \to X$ is a general discrete dynamical system which is compact and point dissipative, then there is a compact global attractor $\mathcal{A}_\pi$ which is connected and Fix(π)$\neq \emptyset$. Theorem 4.1.1 for the scalar 1D equations asserts that the only minimal sets of π are in Fix(π). For the time periodic N-D equation,

$$
\begin{aligned}
u_t - \Delta u &= f(t, x, u, \nabla u) \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} &= 0 \quad \text{in } \partial\Omega
\end{aligned}
\tag{4.1.3}
$$

with $\Omega \subset \mathbb{R}^N$ having a smooth boundary, this property may not hold. A subharmonic solution of (4.1.3) is a solution which is periodic of minimal period $n\omega$ where $n > 1$. Dancer and Hess (1994), Takáč (1991) have shown that there can be stable subharmonic solutons even for the case where the function f in (4.1.3) is independent of ∇u. Poláčik and Tereščák (1991) have shown that the ω_π-limit set of most solutions is a subharmonic solution and these limiting solutions are linearly stable. This is the analogue of most solutions converging to equilibrium in the autonomous case.

Let us assume now that there is a compact global attractor $\mathcal{A}_\pi$ for the Poincaré map π. If we assume that each $\varphi \in \text{Fix}\,(f)$ is hyperbolic, then Theorem 4.1.1 implies that

$$\mathcal{A}_\pi = \cup_{\varphi \in \text{Fix}\,(f)} W_\pi^u(\varphi),$$

where $W_\pi^u(\varphi)$ is the unstable manifold of φ.

If $\varphi \in \text{Fix}\,(\pi)$, then the *Floquet multipliers of* φ, $\lambda_j(\varphi)$, are defined to be the nonzero elements of the spectrum of $\pi'(\varphi)$. The following analogue of the Sturm-Liouville theory for linear second order ODE is due to Chen, Chen and Hale (1992), Chow, Lu and Mallet-Paret (1994) and plays an essential role in the understanding of the global flow defined by π.

Theorem 4.1.2. *If $\varphi \in \text{Fix}\,(\pi)$, then the Floquet multipliers are real and simple, $\lambda_1(\varphi) > \lambda_2(\varphi) > \cdots > \lambda_n(\varphi) > \cdots$, $\lambda_n(\varphi) \to 0$ as $n \to \infty$. Furthermore,* *the corresponding eigenfunction* φ_n *of* λ_n *has only simple zeros in* $(0,1)$ *and* $z(\varphi_n) = n-1$.

Chow, Lu and Mallet-Paret (1994) have shown that the complete Floquet theory holds for linear periodic 1D scalar parabolic equations with Dirichlet boundary conditions; that is, given a linear equation of the form (4.1.1) with Dirichlet boundary conditions, there is a periodic transformation of variables in the function space which takes the equation into a linear 1D parabolic equation with constant coefficients. The proof of this result involves first diagonalizing the system using the Floquet solutions mentioned above and then using inverse scattering theory of Gel'fand, Levitan and Marchenko. Verifcation that the Floquet exponents asymptotically satisfy the conditions of scattering theory involves many nontrivial and remarkable estimates. The diagonalization process is true for much more general dependence on t (see Chow, Lu and Mallet-Paret (1995)). In this latter paper, periodic and Neumann boundary conditions also are discussed.

As in the autonomous case, Theorem 4.1.2 plays an important role in proving the following result of Chen, Chen and Hale (1992).

Theorem 4.1.3. *If* $\varphi \in \mathrm{Fix}\,(\pi)$, $\psi \in \mathrm{Fix}\,(\pi)$ *are hyperbolic, then* $W^u_\pi(\varphi)$ *is transversal to* $W^s_\pi(\psi)$.

Theorem 4.1.3 is fundamental in the proof given by Chen, Chen and Hale (1992) of the Morse-Smale property.

Theorem 4.1.4. *If* π *possesses a compact global attractor and the fixed points of* π *are hyperbolic, then* π *is Morse-Smale and thus structurally stable.*

A point $\varphi \in \mathcal{A}_\pi$ is *chain recurrent* if, for any $\varepsilon > 0$, there are an integer $k \geq 1$ and points $\varphi_0, \varphi_1, \ldots, \varphi_k \in \mathcal{A}_\pi$ such that $|\pi\varphi_i - \varphi_{i+1}| < \varepsilon$ for $0 \leq i < k$. Chen and Poláčik (1995) have proved the following result.

Theorem 4.1.5.

(1) *For any* $\varepsilon > 0$, *there is a Morse decomposition* $\{\mathcal{A}^i_\pi\}_{i\in I}$ *of* $\mathcal{A}_\pi$ *such that*

$$\mathrm{dist}\,(\cup_{i\in I}\mathcal{A}^i_\pi,\ \mathrm{Fix}\,(\pi)) < \varepsilon$$

and each Morse set is either a one point set or a C^1 *one dimensional submanifold with boundary embedded in* X.

(2) *The recurrent set of* π *is* $Fix\,(\pi)$.

(3) π *is gradient-like; that is, there is a continuous Lyapunov function* $V : \mathcal{A}_\pi \to \mathbb{R}$ *such that, for any* $\varphi \in \mathcal{A}_\pi \setminus \mathrm{Fix}\,(\pi)$, $V(\pi\varphi) < V(\varphi)$.

It would be very desirable to have a complete discussion of each of the problems that we considered in Chapter 2 when the vector field is allowed to depend periodically in time. At this time, there are a few general results with the following important one on stability due to Hess (1987).

THEOREM 4.1.6. *If the function f in (4.1.1) is spatially independent and we impose homogeneous Neumann boundary conditions, then the only stable ω-periodic solutions of the system are spatially homogeneous.*

REMARK 4.1.3. For the system (4.1.3) with $f = f(t, u, \nabla u)$ and $\Omega \subset \mathbb{R}^N$ *convex* with a smooth boundary, HESS (1987) has shown that the only stable subharmonic solutions are spatially homogeneous. Since the spatially homogeneous solutions are solutions of the scalar ODE $\dot{u} = f(t, u, 0)$, it follows that the only stable subharmonic solutions are harmonics; that is, ω-periodic.

HESS (1992) has obtained some interesting results for a generalization of the Fisher equation (2.4.1) in several space variables and coefficients periodic in t. Consider the equation

$$
\begin{aligned}
u_t - k(t)\Delta u &= m(x,t)h(u) \quad \text{in}\, \Omega \\
Bu &= 0 \quad \text{in}\, \partial\Omega
\end{aligned}
\tag{4.1.4}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, the boundary operator corresponds to Dirichlet, Robin or Neumann, h is a C^2 concave function (assumed to be strictly concave if B corresponds to Neumann boundary conditions) on an interval I, where either

$$
I = [0, a], \quad h(0) = h(a) = 0,\ h'(0) > 0
$$

or

$$
I = [0, \infty), \quad h(0) = 0, h(s) > 0 \text{ for } s > 0, \frac{h(s)}{s} \to 0 \text{ as } s \to \infty.
$$

The function k is positive Hölder continuous and ω-periodic and $m(x,t)$ is Hólder continuous on $\bar{\Omega} \times \mathbb{R}$ and ω-periodic in t.

The following result is due to Hess (1992). In the statement of the theorem, we exclude the point a when mentioning stability with respect to initial data in I if $I = [0, a]$.

THEOREM 4.1.7. *Each solution of (4.1.4) converges to an ω-periodic solution in $C^2(\bar{\Omega})$ as $t \to \infty$.*

(1) *If $u = 0$ is linearly stable, then the trivial ω-periodic solution is globally asymptotically stable with respect to initial conditions with values in I.*
(2) *If $u = 0$ is linearly unstable, then there is a unique positive ω-periodic solution which is globally asymptotically stable with respect to nontrivial initial conditions with values in I.*
(2) *If $u = 0$ is linearly neutrally stable and $h(s)$ is not linear in some interval $[0, s^*]$ with $s^* > 0$, then $u^* = 0$ is globally asymptotically stable with respect to nonnegative initial data with values in I. If $h(s)$ is linear on $[0, s^*]$ with $s^* > 0$, then there exists a a nontrivial one-parameter family $\{\varepsilon\varphi : 0 \le s \le s^*\}$ of ω-periodic solutions and each solution of (4.1.4) with initial conditions with values in I converges to an element of this family.*

For the case of Neumann boundary conditions, the condition of strict concavity of f is not necessary if $I = [0, a]$. In this case it also is possible to have $m(x, t, u)$ depend upon u if it nonincreasing (see Hess and Weinberger (1990)).

For a further more detailed discussion of periodic parabolic equations as well as many references, see Hess (1991).

4.2. Almost automorphic-separated boundary conditions

The next natural problem to consider is (4.1.1) when the dependence upon time is almost periodic. To obtain a theory as complete as the one for periodic coefficients, we would need to have the analogue of Theorem 4.1.2 which would state that each bounded solution is asymptotic as $t \to \infty$ to an almost periodic function. Unfortunately, this is not even the case for scalar ODE (see Johnson (1981) and the references therein). The appropriate class of functions to consider which will be closed under taking limits as $t \to \infty$ turns out to the almost automorphic functions.

A continuous function $g(t), t \in \mathbb{R}$, is said to be *almost automorphic* if, for any sequence $t_n \to \infty$ as $n \to \infty$, there is a subsequence (which we label the same) and a continuous function $h(t), t \in \mathbb{R}$ such that $g(t + t_n) - h(t) \to 0$, $h(t - t_n) - g(t) \to 0$ pointwise for $t \in \mathbb{R}$ (Bochner (1962)). For any almost automorphic function g, it is possible to define its Fourier series, but it may not be unique and may only converge pointwise in terms of Bochner-Fejer summation. Nevertheless, its frequency module $\mathcal{M}(g)$ can be uniquely defined. We say that a function h is in the hull $\mathcal{H}(g)$ of g if there is a sequence $\{\tau_n\} \subset \mathbb{R}$ such that $\tau_n \to \infty$, $g(t + \tau_n) \to h(t)$ as $n \to \infty$ uniformly for $t \in \mathbb{R}$.

The following remarkable result (new even for ODE) showing that limits of bounded solutions must be almost automorphic and not necessarily almost periodic is due to Shen and Yi (1995). This is the analogue of Theorem 4.1.1 for periodic dependence on time.

Theorem 4.2.1. *If $f(t, x, u, p)$ is almost automorphic in t uniformly with respect to $x \in [0, 1]$ and u, p in bounded sets and $u(x, t)$ is a solution of (4.1.1), (4.1.2) bounded for $t \geq 0$, then there is a sequence $\{\tau_n\}$, $\tau_n \to \infty$ as $n \to \infty$ and an almost automorphic function $v(x, t)$ such that $u(\cdot, t + \tau_n) - v(\cdot, t) \to 0$ as $n \to \infty$ uniformly on compact sets of $\mathbb{R}$ and $v(\cdot, t)$ is a solution of (4.1.1), (4.1.2) with f replaced by some function $g \in \mathcal{H}(f)$. Furthermore, the module $\mathcal{M}(v)$ of the function v is contained in $\mathcal{M}(f)$.*

Remark 4.2.1. There are results for (4.1.3) when the coefficients in time are almost automorphic which are analogous to the ones stated in Remarks 4.1.2, 4.1.3 for the periodic case. In fact, Shen and Li (1995) have shown that, for any almost automorphic function which is the limit of of solutions of (4.1.3) in the sense described in Theorem 4.2.1, there is an integer M such that $M\mathcal{M}(g) \subset \mathcal{M}(f)$. They also have shown that, if $f = f(t, u, \nabla u)$ in (4.1.3) is independent of x, then any linearly stable almost automorphic (almost periodic) solution u of (4.1.3) is spatially homogeneous and is therefore a solution of

$\dot{u} = f(t, u, 0)$. Furthermore, $\mathcal{M}(u) \subset \mathcal{M}(f)$ ($M = 1$ in this case).

4.3. Periodic-periodic boundary conditions

In Section 3, for an autonomous 1D scalar parabolic equation on S^1, we have seen that the flow on limit sets is similar to the flow on the plane. If we allow the coefficients to depend upon t in a periodic way, it might be expected that the flow for the Poincaré map π restricted to limit sets might be similar to the flow on a torus. In some cases, this is true, but the general situation is much more complicated. The following results are due to SANSTEDE and FIEDLER (1992) for the equation

$$u_t = u_{xx} + f(t, x, u, u_x), \quad x \in S^1, \tag{4.3.1}$$

where $f(t, x, u, p)$ is ω-periodic in t and f is a C^2-function. We choose S^1 to be the homeomorphic image of $[0, 1)$. The state space is chosen to be $H^2(S^1)$ and π represents the Poincaré map.

The ω-limit set $\omega(\varphi)$ of a the solution $u(t, \cdot, \varphi)$ of (4.3.1) with initial data φ at $t = 0$ is defined as the set of (τ, ψ) such that there is a sequence $t_k \to \infty$ as $k \to \infty$, such that $\tau = \lim_{k\to\infty} t_k (\mathrm{mod}\, \omega)$, $\psi = \lim_{k\to\infty} u(t_k, \cdot, \varphi)$. The ω-limit set $\omega_\pi(\varphi)$ under the Poincaré map is defined in the usual way.

Let $\sigma_a : H^2(S^1) \to H^2(S^1)$ be the shift map $(\sigma_a\varphi)(x) = \varphi(x + a)$ and $\Sigma\varphi = \{\sigma_a\varphi : a \in S^1\}$.

THEOREM 4.3.1. *Let $f(t, u, u_x)$ be independent of x and ω-periodic in t. If $u(t, \cdot, \varphi)$, $u(0, \cdot, \varphi) = \varphi$, is a solution of (4.3.1) in $H^2(S^1)$ which is uniformly bounded in t, then there is a $\psi \in \omega_\pi(\varphi)$ such that $\omega_\pi(\varphi) \in \Sigma\psi$. Moreover, there is an $\alpha \in S^1$ such that, for each $\xi \in \Sigma\psi$, $\pi\xi = \sigma_\alpha\xi$.*

Since f is independent of the spatial variable x, the map π commutes with any shift σ_a and, therefore, $\pi^n\xi = \sigma_{n\alpha}\xi$. As a consequence, on any ω-limit set of (4.3.1), the action of π is *linear* and the flow is periodic if α is rational and ergodic if α is irrational.

SANSTEDE (1993) has noted that every S^1-equivariant time periodic vector field in the plane embeds by a linear transformation into an S^1-equivariant equation (4.3.1) for a suitable $f(t, u, u_x)$. In fact, any S^1-equivariant vector field in the plane can be written as

$$\dot{y}_1 = f_1 y_1 - f_2 y_2, \quad \dot{y}_2 = f_2 y_1 + f_1 y_2,$$

where $f_j = f_j(y_1^2 + y_2^2)$, $j = 1, 2$. On the other hand, for the PDE

$$u_t = u_{xx} + f_1(u^2 + u_x^2)u + f_2(u^2 + u_x^2)u_x,$$

the transformation $y \mapsto y_1 \sin x + y_1 \cos x$ from $\mathbb{R}^2$ to $V = \mathrm{span}\,(\sin x, \cos x) \subset H^2(S^1)$ is a diffeomorphism which maps orbits onto orbits and preserves sense of direction in time.

If $f(t, x, u, u_x)$ does not possess S^1 symmetry, then the ω-limit sets can be at least as complicated as the ω-limit sets of time-periodically forced vector fields in the plane. In fact, SANSTEDE and FIEDLER (1992) prove

THEOREM 4.3.2. *Every two-dimensional time-periodic vector field in the plane embeds into an equation of the form (4.3.1) for a suitable $f(t, x, u, u_x)$ by means of a linear transformation.*

This result implies that complicated dynamics may occur in parabolic equations on the circle; in particular, one may obtain horseshoes for the Poincaré map by considering say the linearly damped periodically forced pendulum.

In the case of autonomous equations on the circle, the periodic orbits were always unstable and the only stable minimal sets were equilibrium points. It would be interesting to characterize the possible ω-limit sets which can be stable for the time periodic case on S^1. If we introduce nonlocal terms in the equation on S^1, it also should be possible to increase the types of stable ω-limit sets that can occur as we did for the autonomous case in Section 3.4?

REFERENCES

1. ALIKAKOS, N.D., BATES, P. and G. FUSCO (1991). Slow motion for the Cahn-Hilliard equation in one space dimension. *J. Differential Equations* **90**, 81–135.
2. ALIKAKOS, N.D. and G. FUSCO (1992). *Equilibrium and dynamics of bubbles for the Cahn-Hilliard equation.* Preprint.
3. ANGENENT, S. (1988). The zero set of a solution of a parabolic equation. *J. reine angew. Math.* **390**, 79–96.
4. ANGENENT, S. (1986). The Morse-Smale property for a semi-linear parabolic equation. *J. Differential Equations* **62**, 427–442.
5. ANGENENT, S. and B. FIEDLER (1988). The dynamics of rotating waves in scalar reaction diffusion equations. *Trans. AMS* **307**, 545–568.
6. ANGENENT, S., MALLET-PARET, J. and L. PELETIER (1987). Stable transition layers in a semilinear boundary value problem. *J. Differential Equations* **67**, 212–242.
7. ARRIETA, J., CARVALHO, A.N. and J.K. HALE (1992). A damped hyperbolic equation with a critical exponent. *Comm. PDE* **17**, 841–866.
8. BABIN, A.V. and M.I. VISHIK (1991). *Attractors of Evolution Equations*, North Holland.
9. BOCHNER, S. (1962). A new approach to almost periodicity. *Proc. Nat. Acad. Sci.* **48**.
10. BROWN, K.J. and P. HESS (1990). Stability and uniqueness of positive solutions for a semilinear elliptic boundary value problem. *Differential and Integral Equations* **3**, 201–207.
11. BROWN, K.J., LIN, S.S. and A. TERTIKAS (1989). Existence and nonexistence of steady-state solutions for a selection-migration model in population genetics. *J. Math. Biol.* **27**, 91–104.

12. Brunovský, P. (1989). *The attractor of the scalar reaction diffusion equation.* Preprint.
13. Brunovský, P. and B. Fiedler (1989). Connecting orbits in scalar reaction diffusion equations II: The complete solution. *J. Differential Equations* **81**, 106–135.
14. Brunovský, P., Poláčik, P. and B. Sanstede (1992). Convergence in general periodic parabolic equations in one space dimension. *Nonlinear Anal. TMA* **18**, 209–216.
15. Brunovský, P. and P. Poláčik (1997). Morse-Smale structure of generic reaction diffusion equations in higher space dimensions. *J. Differential Equations* **135**, 129–181.
16. Calsina, A. and C. Perelló (1995). Equations for biological evolution. *Proc. Roy. Soc. Edinburgh* **125**, 939–958.
17. Carr, J., Gurtin, M. and M. Slemrod (1984). Structured phase transition on a finite interval. *Arch. Rat. Mech. Anal.* **86**, 317–351. Section 8.2.
18. Carr, J. and R.L. Pego (1989). Metastable patterns in solutions of $u_t = \varepsilon^2 u_{xx} - f(u)$. *Comm. Pure Appl. Math.* **42**, 523–576.
19. Carr, J. and R.L. Pego (1990). Invariant manifolds and metastable patterns in solutions of $u_t = \varepsilon^2 u_{xx} - f(u)$ for small ε. *Proc. Roy. Soc. Edinburgh* **116A**, 133–160.
20. Carvalho, A.N. (1995). Contracting sets and dissipation. *Proc. Roy. Soc. Edinburgh* **125A**, 1305–1329.
21. Carvalho, A.N., Cholewa, J.W. and T. Dlotko (1995). *Examples of global attractors in parabolic problems.* Notas do ICMSC, Série Mat. **29**. Submitted.
22. Carvalho, A.N. and J.A. Cuminato (1997). Reaction-diffusion problems in cell tissues. *J.Dyn.Diff.Eqns* **9**, 93–132.
23. Carvalho, A.N. and A.L. Pereira (1994). A scalar parabolic equation whose asymptotic behavior is dictated by a system of ordinary differential equations. J. Differential Equations **112**, 81–130.
24. Carvalho, A.N. and J.K. Hale (1991). Large diffusion with dispersion. *Nonlinear Anal. TMA* **17**, 1139–1151.
25. Carvalho, A.N., Rodriguez, H.M. and T. Dlotko (1996). *Upper semicontinuity of attractors and synchronization.* Preprint.
26. Casten, R.C. and C.J. Holland (1978). Instability results for reaction diffusion equations with Neumann boundary conditions. *J. Differential Equations* **27**, 266–273.
27. Chafee, N. and E.F. Infante (1974). A bifurcation problem for a nonlinear parabolic equation. *Applicable Anal.* **4**, 17–37.
28. Chafee, N. (l981). The electric ballast resistor: homogeneous and nonhomogeneous equilibria. *Nonlinear Differential Equations* (Eds. P. de Mottoni and L. Salvadori), Academic Press, 97–127.
29. Chafee, N. (1975). Asymptotic behavior for solutions of a one dimensional parabolic equation with Neumann boundary conditions. *J. Differ-*

ential Equations **18**, 111–135.
30. Chen, M., Chen, X-Y. and J.K. Hale (1992). Structural stability for time-periodic one-dimensional parabolic equations. *J. Differential Equations* **96**, 355–418.
31. Chen, X-Y and H. Matano (1989). Convergence, asymptotic periodicity, and finite-point blow-up in one dimensional semilinear parabolic equations. *J. Differential Equations* **78**, 160–190.
32. Chen, X-Y and P. Poláčik (1995). Gradient-like structure and Morse decompositions for time-periodic one-dimensional parabolic equations. *J. Dyn. Diff. Eqns.* **7**, 73–108.
33. Chipot, M. and Hale, J. (1983). Stable equilibria with variable diffusion. *Contemporary Math.* **17**, 209–213.
34. Chow, S.-N., Lu, K. and J. Mallet-Paret (1994). Floquet theory for parabolic differential equations. *J. Differential Equations* **109**, 147-200.
35. Chow, S.-N., Lu, K. and J. Mallet-Paret (1995). Floquet bundles for scalar parabolic equations. *Arch. Rat. Mech. Anal.* **129**, 245–304.
36. Clément, Ph. and Peletier, L.A. (1985). On a nonlinear eigenvalue problem occuring in population genetics. *Proc. Roy. Soc. Edinburgh* **100A**, 85–101.
37. Conway, E., Hoff, D. and J. Smoller (1978). Large time behavior of solutions of systems of reaction diffusion equations, *SIAM J. Appl. Math.* **35**, 1–16.
38. Ciuperca, I.S. (1996). Reaction-diffusion equations on thin domains with varying order of thinness. *J. Differential Equations* **126**, 244–291.
39. Dancer, E. N. and P. Hess (1991). Stability of fixed points for order-preserving discrete-time dynamical systems. *J. Reine Angew. Math.* **419**, 125–139.
40. Dancer, E. N. and P. Hess (1994). Stable subharmonics in reaction diffusion equations. *J. Differential Equations* **104**, 190–200
41. Fan, H. and J.K. Hale (1993). Large time behavior in inhomogeneous conservation laws. *Arch. Rat. Mech. Anal.* **125**, 201–216.
42. Fan, H. and J.K. Hale (1995). Attractors in inhomogeneous conservation laws and parabolic regularizations. *Trans. AMS* **347**, 1239-1254.
43. Fiedler, B. (1994). Global attractors for one-dimensional parabolic equations: sixteen examples. *Tatra Mountains Math. Pub.* **4**, 67–92.
44. Fiedler, B. and J. Mallet-Paret (1989). A Poincaré-Bendixson Theorem for scalar reaction diffusion equations. *Arch. Rat. Mech. Anal.* **107**, 325–345.
45. Fiedler, B. and P. Polàčik (1990). Complicated dynamics on scalar reaction diffusion equations with a nonlocal term. *Proc. Royal Soc. Edinburgh* **115A**, 167–192.
46. Fiedler, B. and C. Rocha (1995). Heteroclinic orbits of semilinear parabolic equations. *J. Differential Equations.*
47. Fiedler, B. and C. Rocha (1996). Orbit equivalence of global attractors of semilinear parabolic differential equations. Preprint.

48. Fife, P. C. (1974). Transition layers in singular perturbation problems. *J. Differential Equations* **15**, 77–105.
49. Fife, P. C. (1976). Boundary and transition layer phenomena for pairs of second order equations. *J. Math. Anal. Appl.* **54**, 497–521.
50. Fife, P. C. and L. A. Peletier (1981). Clines induced by variable selection and migration. *J. Math. Anal. Appl.* **54**, 99–123.
51. Fleming, W. H. (1975). Aselection migration model in population genetics. *J. Math. Biology* **2**, 219–233.
52. Freitas, P. (1994). Bifurcation and stability of stationary solutions of nonlocal scalar reaction diffusion equations. *J. Dyn. Diff. Eqns.* **6**, 613–630.
53. Freitas, P. (1993). *Stability of stationary solutions for a scalar nonlocal reaction-diffusion equation*. Preprint.
54. Fujii, H. and M. Nishiura (1987). Stability of singularly perturbed solutions to systems of reaction diffusion equations. *SIAM J. Appl. Math.* **18**, 1726–1770.
55. Fusco, G. (1987). A system of ODE which has the same attractor as a scalar ODE. *J. Differential Equations* **69**, 85–110.
56. Fusco, G. (1990). A geometric approach to the dynamics of $u_t = \varepsilon^2 u_{xx} + f(u)$ for small ε. *Lect. Notes Physics* **359**, 53–73.
57. Fusco, G. and J.K. Hale (1985). Stable equilibria in a scalar parabolic equation with variable diffusion. *SIAM J. Math. Anal.* **16**, 1152–1164.
58. Fusco, G. and J.K. Hale (1989). Slow motion manifolds, dormant instability and singular perturbations. *J. Dyn. Diff. Equations* **1**, 75–94.
59. Fusco, G., Hale, J.K. and J. Xun (1996). Traveling waves as limits of solutions on bounded domains. *SIAM J. Math. Anal.* **27**, 1544–1588.
60. Fusco, G. and W. Oliva (1988). Jacobi matrices and transversality. *Proc. Roy. Soc. Edinburgh* **109A**, 231–243.
61. Fusco, G. and W. Oliva (1990). Transversality between invariant manifolds of periodic orbits for a class of monotone dynamical systems. *J. Dyn. Diff. Eqns.* **2**, 1–17.
62. Fusco, G. and C. Rocha (1991). A permutation related to the dynamics of a scalar parabolic PDE. *J. Differential Equations* **91**, 75–94.
63. Hale, J. K. (1986). Large diffusivity and asymptotic behavior in parabolic systems, *J. Math. Anal. Appl.* **118**, 455–466.
64. Hale, J.K. (1988). *Asymptotic Behavior of Dissipative Systems*, Am. Math. Soc. Providence.
65. Hale, J.K. (1997). Diffusive coupling, dissipation and synchronization. J. Dyn. Diff. Eqs. **9**, 1–52.
66. Hale, J.K., LaSalle, J.P. and M. Slemrod (1972). Theory of a general class of dissipative processes. *J. Math. Anal. Appl.* **39**, 177–191.
67. Hale, J.K., Magalhães, L. and W. Oliva (1984). *An Introduction to Infinite Dimensional Dynamical Systems - Geometric Theory*, Springer-Verlag.
68. Hale, J.K. and G. Raugel (1992). Attractors for dissipative evolution-

ary equations. In *Equadiff 91, International Conf. on Differential Equations, Barcelona, 1991,* World Scientific.

69. Hale, J.K. and G. Raugel (1992). Convergence in gradient-like systems and applications. *ZAMP* **43**, 63–124.
70. Hale, J.K. and G. Raugel (1992). Reaction-diffusion equations on thin domains. *J. Math. Pures Appl.* **71**, 33–95.
71. Hale, J.K. and C. Rocha (1985). Bifurcation in a parabolic equation with variable diffusion. *Nonlinear Anal. TMA* **9**, 479–494.
72. Hale, J.K. and C. Rocha (1987). Varying boundary conditions with large diffusivity. *J. Math. pures appl.* **66**, 139–158.
73. Hale, J.K. and C. Rocha (1987). Interaction of diffusion and boundary conditions. *Nonlinear Anal. TMA* **11**, 633–649.
74. Hale, J.K. and K. Sakamoto (1989). Shadow systems and attractors in reaction diffusion equations. *Applicable Anal.* **32**, 287–303.
75. Hale, J.K. and K. Sakamoto (1988). Existence and stability of transition layers. *Japan J. Appl. Math.* **5**, 367–405.
76. Hale, J.K. and J. Vegas (1984). A nonlinear parabolic equation with varying domain. *Arch. Rat. Mech. Anal.* **2**, 99–123.
77. Hale, J.K. and S. Verduyn Lunel (1993). *Introduction to Functional Differential Equations*, Appl. Math. Sci., **99**, Springer.
78. Haraux, A. (1989). Systèmes Dynamiques Dissipatifs et Applications. *Res. Math. Appl. (RMA).* **17**, Masson.
79. Henry, D. (1981). *Geometric Theory of Semilinear Parabolic Equations.* Lect. Notes Math. **840**, Springer-Verlag.
80. Henry, D. (1985). Some infinite dimensional Morse-Smale systems defined by parabolic differential equations. *J. Differential Equations* **59**, 165–205.
81. Hess, P. (1987). Spatial homogeneity of stable solutions of some periodic problems with Neumann boundary conditions. *J. Differential Equations* **68**, 320–331.
82. Hess, P. (1991). Periodic-parabolic boundary value problems and positivity. *Pitman Res. Notes Math.* **247**, Longman.
83. Hess, P. (1992). Asymptotics in semilinear periodic diffusion equations with Dirichlet and Robin boundary conditions. *Arch. Rat. Mech. Anal.*
84. Hess, P. and H. Weinberger (1990). Convergence to spatial-temporal clines in the Fisher equation with time-periodic fitness. *J. Math. Biol.* **28**, 83–98.
85. Hirsch, M.W. (1983). Differential equations and convergence almost everywhere in strongly monotone semiflows. *Contemp. Math.* **17**, 267–285.
86. Hirsch, M.W. (1988). Systems of differential equations which are competitive or cooperative: III, Competing species. *Nonlinearity* **1**, 51–71.
87. Hirsch, M.W. (1988). Stability and convergence in strongly monotone dynamical systems. *J. Reine Angew. Math.* **383**, 1–58.
88. Hoppensteadt, F.C. (1975). Analysis of a stable polymorphism arising in a selection-migration model in population genetics dispersion and

selection. *J. Math. Biol.* **2**, 235–240.

89. Ito, M. (1984). A remark on singular perturbation methods. *Hiroshima Math. J.* **14**, 619–629.
90. Johnson, R.A. (1981). Bounded solutions of scalar almost periodic linear equations. *Illinois J. Math.* **15**.
91. Jolly, M.S. (1989). Explicit construction of an inertial manifold for a reaction diffusion equation. *J. Differential Equations* **78**, 220–261.
92. Jones, C. (1984). Stability of traveling wave solution of the Fitzhugh-Nagumo equation. *Trans. Am. Math. Soc.* **286**, 431–469.
93. Kruzkov, S.N. (1970). First order quasilinear equations in several independent variables. *Mat. Sb.* (N.S.) **81**, 228–255; English translation, *Math. USSR-Sb.* **10**, 217–243.
94. Kurland, H. (1983). Monotone and oscillatory equilibrium solutions of a problem arising in population genetics. *Contemporary Mathematics* **17**, 323–342, Am. Math. Soc., Providence, RI.
95. Ladyzenskaya, O.A. (1987). On the determination of the minimal global attractors for the Navier Stokes equation and other differential operators. *Russian Mathematical Surveys* **42**, 27–73.
96. Levin, S. S. and L. A. Segal (1982). Models of the influence of predation on aspect diversity in prey populations. *J. Math. Biology* **14**, 253–285.
97. Levin, S. S. and L. A. Segal (1985). Pattern generation in space and aspect. *SIAM Review* **27**, 45–67.
98. Lyberopoulos, A.N. (1990). Asymptotic oscillations of solutions of scalar conservation laws with convexity under the action of a linear excitation. *Quart. Appl. Math.* **48**, 755–765.
99. Lyberopoulos, A.N. (1994). Large time structure of solutions of scalar conservation laws with a nonlinear source field. *Proc. Roy. Soc. Edinburgh.*
100. Mallet-Paret, J. (1976). Negatively invariant sets of compact maps and an extension of a theorem of Cartwright. *J. Differential Equations* **22**, 331–348.
101. Mallet-Paret, J. and H.L. Smith (1990). The Poincaré-Bendixson theorem for monotone cyclic feedback systems. *J. Dyn. Diff. Eq.* **2**, 367–421.
102. Mané, R. (1981). On the dimension of the compact invariant sets of certain nonlinear maps. *Lect. Notes Math.* **898**, 230–242.
103. Massatt, P. (1986). *The convergence of scalar parabolic equations with convection to periodic solutions.* Preprint.
104. Matano, H. (1978). Convergence of solutions of one-dimensional semilinear parabolic equations. *J. Math. Kyoto Univ.* **18**, 224–243.
105. Matano, H. (1979). Asymptotic behavior and stability of solutions of semilinear diffusion equations. *Publ. Res. Inst. Math. Sci.* **15**, 401–458.
106. Matano, H. (1982). Nonincrease of lap number of a solution for a one-dimensional similinear parabolic equation. *J. Fac. Sci. Univ. Tokyo Sect. 1A Math.* **29**, 401–441.
107. Matano, H. (1984). Existence of nontrivial unstable sets for equilibriums

of strongly order-preserving systems. *J. Fac. Sci. Univ. Kyoto* **30**, 645–673.

108. Matano, H. (1988). Asymptotic behavior of solutions of semilinear heat equations on S^1. *Nonlinear Diffusion Equations and Their Equilibrium S tates II* (Eds. Ni, Peletier, Serrin), Springer-Verlag, New York, 139–162.
109. Mimura, M., Tabata, M. and Y. Hosono (1980). Multiple solutions of two point boundary value problems of Neumann type with a small parameter. *SIAM J. Math. Anal.* **11**, 613–631.
110. Morita, Y. (1990). Reaction-diffusion systems in nonconvex domains: Invariant manifold and reduced form. *J. Dyn. Diff. Equations* **2**, 69–115.
111. Nascimento, A. (1983). Bifurcation of radially symmetric equilibria of a parabolic equation. *Nonlinear Analysis* **7**, 1061–1070.
112. Nascimento, A. (1990). Nonconstant stable radial distribution of a reaction diffusion equation. *Seminário Brasileiro de Análise. ATAS* **30**, 349–364.
113. Nickel, K. (1962). Gestaltaussagen über Lösungen parabolischer Differentialgleichungen. *J. Reine Angew. Math.* **211**, 78–94.
114. Nishiura, Y. (1982). Global structure of bifurcating solutions for reaction diffusion systems. *SIAM J. Math. Anal.* **13**, 555–593.
115. Oliva, S. (1995). Reaction-diffusion systems on domains with thin channels. *J. Differential Equations* **123**, 437–479.
116. Peletier, L.A. (1976). On a nonlinear equation arising in population genetics. *Proc. 4th Conference in Ordinary and Partial Differential Equations at Dundee*, Lect. Notes Math. **564**, Springer.
117. Peletier, L.A. (1978). A nonlinear eigenvalue problem occuring in population genetics. *Proc. 'Journée d'Analyse nonlinéaire de Besançon'*, Lect. Notes Math. **665**, 170–187, Springer.
118. Pinto, J. (1995). *Slow motion manifolds for reaction diffusion equations.* Ph.D. Thesis, Georgia Tech.
119. Poláčik, P. (1991). Complicated dynamics in scalar semilinear parabolic equations in higher space dimension. *J. Differential Equations* **89**, 244–271.
120. Poláčik, P. (1995). High-dimensional ω-limit sets and chaos in scalar parabolic equations. *J. Differential Equations* **119**, 24–53.
121. Poláčik, P. (1994). Transversal and nontransversal intersection of stable and unstable manifolds of reaction diffusion equations on symmetric domains. *Differential and Integral Equations* **7**, 1527–1545.
122. Poláčik, P. and K. Rybakowski (1995). Imbedding vector fields into Dirichlet BVP's. *Ann. Scuola Norm. Sup. Pisa.* **21**(4), 737–749.
123. Poláčik, P. and K. Rybakowski (1996). Nonconvergent bounded trajectories in semilinear heat equations. *J. Differential Equations* **124**, 472–494.
124. Poláčik, P. and I. Tereščák (1991). Convergence to cycles as a typical asymptotic behavior in smooth strongly monotone discrete-time dynamical systems. *Arch. Rat. Mech. Anal.* **116**, 339–360.
125. Rocha, C. (1985). Generic properties of equilibria of reaction diffusion

equations with variable diffusion. *Proc. Roy. Soc. Edinburgh* **101A**, 45–56.

126. Rocha, C. (1988). Examples of attractors in scalar reaction diffusion equations. *J. Differential Equations* **73**, 178–195.
127. Rocha, C. (1991). Properties of the attractor of a scalar parabolic PDE. *J. Dyn. Diff. Eqns.* **3**, 575–592.
128. Rocha, C. (1994). Bifurcations in discretized reaction-diffusion equations. *Resenhas IME-USP* **1**, No. 4, 1–16.
129. Sanstede, B. (1993). Asymptotic behavior of solutions of non-autonomous scalar reaction-diffusion equations. *Int. Conf. Differential Equations, Barcelona 1991* (Eds. Perelló, Simó, Solà-Morales), World Scientific.
130. B. Sanstede and B. Fiedler (1992). Dynamics of periodically forced parabolic equations on the circle. *Ergod. Theory Dyn. Sys.* **12**, 559–571.
131. Shen, W. and Y. Yi (1995). On minimal sets of scalar parabolic equations with skew-product structure. *Trans. Am. Math. Soc.* To appear
132. Shen, W. and Y. Yi (1995). *Almost automorphy and skew-product semiflow*. Parts II, III. Preprint.
133. Smith, H. (1995). *Monotone Dynamical Systems*. Am. Math. Soc.
134. Takáč, P. (1991). *A construction of stable subharmonic orbits in monotone time-periodic dynamical systems*. Preprint.
135. Taniguchi, M. (1996). A remark on singular perturbation methods via the Lyapunov-Schmidt reduction. *Pub. Res. Inst. Math. Sci.*
136. Temam, R. (1998). *Infinite Dimensional Dynamical Systems in Mechanics and Physics*. Springer.
137. Valero, J. (1967). *Existence and dimension of attractors in dynamical systems generataed by differential inclusions*. Ph.D. Thesis, Universidad de Murcia, Spain.
138. Vol'pert, A.I. (1967). The space BV and quasilinear equations. *Mat. Sb.* (N.S) **73**; English translation, *Math.-Sb.* **2**, 225–267.
139. Yanagida, E. (1982). Stability of stationary distribution in space- dependent population growth processes. *J. Math. Biology* **15**, 401–441.
140. Zelenyak, T.I. (1968). Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable. *Differential Equations* **4**, 17–22.